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Doctoral Thesis from the Department of  
Mathematics and Science Education 22

# Individualized Mathematics Instruction for Adults

The Prison Education Context

Linda Marie Ahl



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Linda Marie Ahl

Academic dissertation for the Degree of Doctor of Philosophy in Mathematics Education at Stockholm University to be publicly defended on Friday 8 May 2020 at 13.00 in Hörsal 7, hus D, Universitetsvägen 10 D.

### Abstract

Individualized instruction tailors content, instructional technology, and pace to the abilities and interests of each student. Carrying out individualized instruction for adults returning to mathematics after some years away from schooling entail special challenges. Adults have, to a greater extent than children and adolescents, various prior knowledge from former schooling. Their rationales for learning mathematics differ from children and adolescents. The main triggers for adults to study mathematics are to get qualification for further studies; to prove that they can succeed in a subject where they have previously experienced failure; to help their children and to experience understanding and enjoyment. Adults also struggle with negative affective feelings against mathematics as a subject and with mathematics anxiety to a greater extent than children and adolescent learners.

Much is known about the special challenges in teaching adults but less is known of how to adapt this knowledge into teaching practice. This thesis addresses the aim of how to organize individual mathematics instruction for adult students without an upper secondary diploma, so that they are given opportunities to succeed with their studies and reach their individual goals.

In the context of the Swedish prison education program four case studies were conducted to address the aim. The methods used were: development and evaluation of a student test of prior knowledge on proportional reasoning combined with clinical interviews; interviews focusing on a student's rationales for learning; a retrospective analysis of events in relation to feedback situations; an analysis of a common student error in relation to the role of language representation as a signifier for triggering students' schemes.

The results showed, first, that the test together with the clinical interview elicited students' prior knowledge on proportional reasoning well and that different students could be classified in qualitatively different ways. Second, that the theoretical construct of instrumental- and social rationales for learning was useful for understanding a student's initial and changing motivation in relation to the teaching and to the practice of mathematics the teaching entails. Third, that a delay between written and oral feedback worked as a mechanism that gave the receiver time and space to reflect on the feedback, which led to circumventing situations where the student ended up in affect that hindered him from receiving the teacher's message. Forth, that a linguistic representation in the problem formulation led to a common error, triggering two separate schemes. As a result of the analysis, a theoretical extension of Vergnaud's theory was suggested by detailing the relationship between schemes and semiotics.

The results are transformed into a model for individualized mathematics instruction of adults, MIMIA, in the Swedish prison education program. MIMIA consist of a flowchart for using practical- and thinking tools for individualizing instruction. The practical tools are used to elicit students' prior knowledge and organize feedback situations for adults with negative affective feelings towards mathematics. The thinking tools are used to understand and classify adult students' rationales for learning and to analyze students' solution schemes in relation to language representations in the problem statements.

**Keywords:** *Individualized Instruction, Mathematics, Prison Education, Adults, Tutoring.*

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# List of Papers

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# 1 Introduction

Since 2007 I have been teaching mathematics in the Swedish prison education program. Between the years of 2012 and 2014 I attended a graduate school for teachers, leading to a licentiate degree in didactics of mathematics (mathematics education). My focus at the time was curriculum materials; the representation of proportional reasoning in textbooks and educative features in teacher guides. When I returned to full time teaching in the prison education program, I became interested in reflecting on my own practice. Before the graduate school I did not have the tools to theoretically analyze, address and seek new knowledge about difficulties and dilemmas in my teaching practice. But now I could, and I did.

All the studies in this thesis are problem driven, sprung from my own experiences in my teaching practice. I seek to improve my practice and give the best possible conditions for my students. This is not only a matter of professional development, it is also a research endeavor and this thesis is a product of this endeavor. As a researcher on my own practice I follow a long and growing tradition of researchers aiming to produce knowledge that can contribute to the improvement of teaching and learning close to the context that they work in (c.f. Anderson, 2002; Ball, 2000; Lampert, 2000; Mercer, 2007; Teusner, 2016; Wilson, 1995). The four studies I have carried out are of different kinds, unified by the interest to gain knowledge about individualized mathematics instruction of adults in the special context of the Swedish prison education program.

## 1.1 Context

The Swedish prison and probation service implemented a new prison education program in 2008. The new organization, with around 110 teachers spread across Sweden's 45 prisons and some custodies, can offer approximately 130 different upper secondary courses and thus offer prisoners a range of possibilities, from single courses to a complete upper secondary diploma. Adult education within the Prison and Probation Service is organized in special facilities, so called Learning Centers. The organization with teachers at each prison, and some custodies, makes education available all over the country. A distance education model makes all courses available for all prisoners meaning that

teachers teach both locally and on distance. The distance education system relies on an intranet where students and teachers can communicate through written messages. Oral communication is provided over the phone. The distance education model makes it possible for students to retain his or her teacher in the event of any movement between prisons.

The curriculum and syllabuses in the prison education program are the same as for adult education in society. An important condition for the organization of studies is that students in prison need to be able to enroll in courses at any point in time of the year. Education is planned under the conditions that prisoners might start serving their time at any point and that education shall be synced with other planned activities and programs in the Swedish prison and probation service. Therefore, conventional classroom teaching with courses running over a fixed period of time is not an option.

All teaching inside Swedish prisons shall be individualized. This condition is clearly stated in the guidelines for the adult education in prison, established in *Kriminalvårdens handbok för vuxenutbildning* [The Prison and Probation Service manual for adult education] (2018:14). Individualized instruction refers to teaching that tailors content, instructional technology, and pace to the abilities and interests of each student. However, how to organize a teaching that adapts the contents to each student is not described.

## 1.2 Individualized Instruction

A way of organizing individualized instruction is by following the principles of *tutoring*. The concept of tutoring can embrace different meanings. The most common interpretation is that a tutor is an adult, subject-matter expert working synchronously or asynchronously with a single student (Bloom, 1984). Other interpretations of tutoring refer to different kinds of settings where a more or less skilled peer or parent supports one or a small group of students. Also, computer systems, building on learning trajectories where one level must be mastered before getting access to the next level, are referred to by the term tutoring (Anderson, Corbett, Koedinger, & Pelletier, 1995).

With the teaching model one-to-one tutoring, or one-to-one instruction, the tutor or teacher can pursue a given topic or problem until the students have mastered it. (Cohen, Kulik, & Kulik, 1982; Bloom, 1984). Tutoring can be carried out successfully by carefully structuring confirmatory feedback and giving additional guidance when needed (Merrill, Reiser, Merrill, & Landes, 1995). In prison education, the mathematics teacher can carry out this aspect of tutoring, either locally or in a distance setting. Each student that is enrolled on a distance education course has a local teacher as a supervisor, who typically does not teach the subject. Although they are not able to give professional instruction on the subject, they can support and prompt the student,

which can also lead to effective tutoring (Chi, Siler, Jeong, Yamauchi, & Hausmann, 2001).

The individualized instruction in the prison education program gives opportunities for teachers to work with certain content until the student masters the content. The idea of mastering certain content before moving on comes from the teaching strategy and educational philosophy *mastery learning*. Mastery learning builds on students having to achieve a level of mastery of the subject matter in an instructional unit before moving onto the next instructional unit (Kulik, Kulik, & Bangert-Drowns, 1990). The teaching strategy is developed from Bloom's educational approach, Learning for Mastery (Bloom, 1968), with influences from Keller's Personalized System of Instruction (Keller, 1968). The mastery learning methods suggest that the focus of instruction should be the time required for different students to learn the same content and achieve the same level of mastery (Bloom, 1984; Kulik et al., 1990). In conventional teaching, the teacher largely determines the instructional pace and students are required to keep up with the pace if they are to learn effectively. In contrast, mastery learning builds on a shift in responsibilities (Bloom, 1981). When a student fails to learn it is assumed to be due to differences in the student's learning process, rather than because of the student's lack of ability. When a student does not achieve mastery of a given educational unit they are given additional instruction and learning support in cycles, with instructions and tests until mastery is achieved. Mastery learning highly depends on formative strategies (Bloom, 1984). Formative tests are given for feedback followed by teaching addressing the elicited lack of knowledge or misconceptions.

Studies on the tutoring of adults are rare, but studies in grades 5 and 8, have showed that one-to-one tutoring was two standard deviations (2-sigma) more effective than conventional teaching (Bloom, 1984). These results have been criticized and later studies have not been able to replicate the same effectiveness for tutoring (VanLehn, 2011). VanLehn made a review of 44 studies concerning tutoring of students of different ages. Only studies on synchronous tutoring were included in the review:

To match common beliefs about maximally effective forms of human tutoring, studies of human tutoring were restricted to synchronous human tutoring. Synchronous means that there are no long delays between turns, as there are with e-mail or forums. Face-to-face tutoring qualifies as synchronous, and most studies of human tutoring in the review used face-to-face tutoring. (VanLehn, 2011, p. 205)

In VanLehn's review of the effectiveness of human tutoring, one-to-one tutoring still comes out as far more effective than conventional teaching with an effect size of 0.79 standard deviations, even though the 2-sigma result from

Bloom (1984) was not repeated. Tutoring works especially well for the weakest group of learners. The top 20% of students seem to achieve good results in any educational settings (Bloom, 1984). Tutoring have also shown to have positive effects on students' attitudes towards the subject matter covered (Cohen et al., 1982).

Mastery learning has also proved to outperform conventional teaching and has been found to have positive effects on students' attitudes towards course content (e.g., Guskey, & Pigott, 1988; Kulik et al., 1990; Slavin, 1987). However, the studies on the efficacy of mastery learning has so far failed to reach a consensus on a quantitative measure of how large improvements one could expect compared to conventional teaching. However, the different results are not surprising since the studies differed in the subject area to which mastery learning was applied, the grade level of students involved, and the duration of the studies varied. Yet, these studies show that mastery learning consistently yields positive effects on both cognitive and affective learning outcomes. Unlike one-to-one tutoring, mastery learning was originally designed to be implemented in similar learning settings as conventional instruction where students are learning a specific subject matter in a class with about 30 students. The basic idea is that students are given time and instructional support to achieve mastery off a unit before moving on to next. While most studies concern children, the area of teaching adults using tutoring and mastery learning is underdeveloped. However, there have been studies with positive outcomes. For example, Gill and O'Donoghue (2006) has shown that support tutorials can be helpful for addressing the mathematics needs of adult returners.

In summary, it can be argued that individualized instruction that tailors content, instructional technology and pace to the abilities and interests of each student is not a straightforward and unproblematic endeavor. Individualization needs information of each student's interests and abilities. To gain information and adapt it to instruction of each student is the core of individualized instruction. The central problem for this thesis is how to gain information of the students and how to use that information to individualize instruction for the students in the prison education program.

Having now presented my research interest, it is time for a clarification to the reader. Prison teaching often tickles the imagination. It is not uncommon for people I meet to reveal that they have many prejudices about both prisoners and the teaching profession in prison. The question of power relations, threats and violence is often brought up. Therefore, I want to clarify some important aspects of teaching practices in prison. Teachers are a civil function, like priests and psychologists. We wear civil clothes, not uniforms signaling power. Our role is to provide education to adult students, not to incarcerate prisoners. Therefore, this thesis is not about power relations.

In this thesis, instruction is foregrounded and the imprisonment is a backgrounded context. Of course, context is always important for how education can be carried out but my goal is not to problematize the prison institution. I

focus on instruction. In this thesis I claim that individualized mathematics instruction of adults in the Swedish prison education program can improve, if you systematically address certain aspects of teaching and learning. It may seem vain, but my sincere hope is that the insights I share will give more students in the Swedish prisons education program opportunities to complete their mathematics courses with passing grades.

## 1.3 Thesis Aim and Research Questions

The aim of this study is to gain knowledge of how to organize individual mathematics instruction for adult students, without an upper secondary diploma, so that they are given opportunities to succeed with their studies and reach their individual goals. To fulfill the thesis aim, four separate studies have been conducted addressing four separate research questions. The three first questions addresses concerns for adult education in general; the fourth question is a spinoff from the results of the study presented in paper 1.

1. How can adult students' prior knowledge be identified in terms of how developed their proportional reasoning skills are?
2. How can adult students' motivation for studying mathematics in relation to their social context and their mathematical learning environment be characterized?
3. How can the timing of feedback impact adult students with negative affective feelings towards mathematics?
4. How can the signifying role of language representations for triggering erroneous schemes in situations involving scientific concepts be theorized?



## 2 Previous Research

This thesis touches on several research fields. First, I present research on prison education in general as well as some of the few studies on the teaching and learning of the subject mathematics in prison, section 2.1. Then, in section 2.2, I widen my review to adults' mathematics learning in general. These themes form a general background for my work. But, since my studies also embrace motivation and formative feedback I also review research on motivation in section 2.3 and timing of feedback in section 2.4.

### 2.1 Prison Education

Prison education is not a uniform activity. In the report *Prison Education and Training in Europe—A Review and Commentary of Existing Literature, Analysis and Evaluation*, an overview of prison education in Europe is presented (Costelloe & Langelid, 2011). The definition of education used in the report embrace all sorts of educational activities in prison, not only education in the traditional sense, but also addiction studies and cognitive-behavioral programs.

This wide definition is a consequence of a fundamental issue raised within the literature on prison education; namely how to organize it and what the content shall be (Costelloe & Langelid, 2011). The question is whether prison education shall consist of solely classroom-based learning or if the focus shall lay on acquiring skills outside of fixed curriculums. Furthermore, the issue of voluntary or obligatory participation causes discussion. Compulsory education may seem to be an effective way of encouraging participation. However, compulsory education could also have a negative effect on students' motivation for the prisoners that have negative experiences of formal education. Compulsory education would also be in conflict with the fundamental principle and basic premise of adult education; that it is voluntary for adults in society to attend education.

The models for prison education in Europe differ in both content and organization (Costelloe & Langelid, 2011). It is either organized by the educational authorities, the prison and probation service or some combination of both (Karsikas, et al., 2009). Sweden earlier employed a contract model where all educational activities were purchased from educational providers outside

prison (Svensson, 2009). But, since year 2008 all education for prisoners is regulated in legislation, both for the prison and probation and the educational services. Each prison region has a headmaster, who is directly responsible for education in his or her region. The headmasters form a network, coordinated by a national prison education coordinator. In the current model regional headmasters are directly responsible for the teachers who are employed by the Prison and Probation Service.

Studies on prison education often focus on power relations and motivation. Power relations in prison have been studied from various perspectives. For example: the multiple and complex power relations that shape young adults in prison (Mertanen & Brunila, 2018); students' positioning in relation to their own experiences (Tett, 2019); teachers negotiation power relations for democratic classrooms (Spaulding, Banning, Harbour, & Davies, 2009); how the power of the prison regime affect teaching practice (Lukacova, Lukac, Lukac, Pirohova, & Hartmannova, 2018); black male behavioral responses in disempowering educational settings (Dancy, 2014).

Studies on motivation showed that many Irish prisoners view education in prison as a second chance opportunity (Costelloe, 2003). The imprisonment itself created motivation to study to escape from boredom in everyday prison life. Also, motivational factors independent of the imprisonment were reported, like improving employment prospects and making their families proud. Similar results are reported for Greek prison students, by Papaioannou, Anagnou and Vergidis (2018). In Flanders, a study of 486 prison students showed that the strongest motivational factors were the desire to learn, obtaining a degree and making plans for the future (Halimi, Brosens, De Donder, & Engels, 2017). A study on the entire Norwegian prison population confirms the results from the Irish study. With a response rate of 71.1 % three distinct motivational factors were found: preparation for life after release, social reasons related to the prison context, and to acquire formal knowledge and skills (Diseth, Eikeland, Manger, & Hetland, 2008). For imprisoned students in the Nordic countries the most important motivational factor is to spend the prison time doing something sensible and useful (Manger, Eikeland, Diseth, Hetland, & Asbjørnsen, 2010). The relationship between prisoners' academic self-efficacy and participation in education was investigated in a study of Norwegian prisoners (Roth, Asbjørnsen, & Manger, 2016). The authors conclude that participation in education had a positive influence on self-efficacy in both mathematics and self-regulated learning. The prisoners' academic self-efficacy was analyzed via self-reported data. No comparison with actual mastery was done. However, although the prison stay can be a motivating factor in itself, it is important to remember that this initial motivation is usually only enough to get prisoners to enroll in education. Once in the classroom, the initial motivation needs to be maintained by the teacher's instruction (Costelloe & Langvik, 2011).



Studies on the subject of mathematics are rarer. Few journal articles address mathematics teaching and learning in prison. For example, a case study from Finland investigated a prisoner, as a representative for a group of Finnish adults with poor basic mathematics and numeracy skills (Hassi, Hannula, & Saló i Nevado, 2010). Data was collected through an interview. Data showed that all students are given the same instruction, regardless of their prior mathematical knowledge. The prisoners' skills varied a lot. The qualification required to enroll in the mathematics course was rather low, in the interviewee's opinion. Also, he found it hard to be challenged by the instructional material in the course. The prior knowledge and skills of prisoners in England was investigated by Creese (2016). He concludes that the mathematics skill levels of prisoners in England in 2015 were better than the prisoners' skills in the previous assessment in 2011, although lower than in the population in general. These are examples of studies that touches on the mathematics teaching and learning in prison without actually conduct teaching experiments on how to organize teaching. Because of the sparse representation of studies on teaching and learning of mathematics for adults in prison I widened my search to the research field of adults learning mathematics in general.

## 2.2 Adults Mathematics Learning

The research domain of adults' mathematics learning is multifaceted and cultivated between adult education and mathematics education (FitzSimons, & Godden, 2000; Wedege, 2010). The number of studies on adults' mathematics learning is limited, with few articles in mainstream mathematics journals to find on the subject:

Research in adult mathematics education is reported in a disparate variety of publications. A small number of articles have appeared in mainstream mathematics education journals but the overwhelming majority of reports lie hidden in doctoral dissertation research and the proceedings and journal of Adults Learning Mathematics – a Research Forum (ALM). (Safford-Ramus, 2017, p. 28)

The field of study comprises a broad range of settings for research, teaching and learning, which embraces both formal settings with a fixed curriculum, non-formal settings like workplace skills and informal learning, relating to mathematic in life experiences (Evans, Wedege & Yasukawa, 2012). Within this field, some special research interests can be discerned, namely informal-, non-formal-, and formal mathematics learning. While formal learning is the focus of this thesis I briefly mention non-formal and informal mathematics learning before giving an overview of what is known from formal mathematics learning of adults.

While the informal learning perspective concerns situations where some mathematics should be learned, but where the learning situations are not formalized (Evans et al., 2013), non-formal mathematics relates to different workplace skills. In the adult mathematics education literature, the non-formal perspective has been dealt with in a multitude of ways, often involving the issue of transfer as well as the issue of uncovering mathematics in workplace practices that are hidden in so called black boxes (Williams & Wake, 2007). The term black box denotes a procedure, tool, machine etc. whose functioning depends on some mathematics in such a way that the mathematics is not visible to the user. Even though it can be shown that mathematical skills can evolve in work practice it cannot be taken for granted that these skills can be transferred into a formal school context, as shown by Carraher, Carraher, and Schliemann (1985).

Transfer the other way around, from school to work, has also been investigated and has shown to be complicated. Since workplace mathematics is contextualized in particular ways, formal school training might not prepare workers appropriately (Wedge, 2010). An example is the study by Marks, Hodgen, Coben, and Bretscher (2016), which analyzed nursing practices where the accuracy of calculation can often be the difference between life or death. They showed that the numeracy taught in the university often build on very different methods compared to what is used in practice. Furthermore, the assessment situations differ radically from the practical context where the nurses have to perform. This result is also supported by Hoyles, Noss, & Pozzi, (2001).

In studies on formal mathematics education for adults a recurring theme is that adult learners struggle with negative affective feelings against mathematics as a subject and with mathematics anxiety to a greater extent than children and adolescent learners (Wedge & Evans, 2006; Klinger, 2011; Ryan, & Fitzmaurice, 2017). Beliefs, attitudes, and emotions are used to describe a wide range of affective responses to mathematics (McLeod, 1992), such as positive or negative preferences, attitudes, emotions and moods. Schlöglmann (2006) concludes that “Mathematics in particular is often associated with negative memories, and so people try to avoid using mathematics in their everyday or vocational lives. This leads to a problematic affective situation in adult-educational mathematics courses.” (p. 15) Two different studies trying to address the problem with adults’ negative affective feelings towards mathematics have used students writing (Hauk, 2005; Viskic & Petocz, 2006), for both data collection and as a tool for students’ self-regulation and awareness. In the study by Hauk (2005) 67 autobiographical essays from students in a college liberal arts mathematics course was examined. She found that reflective writing could support students’ self-efficacy. Written reflections to investigate adult university students’ ideas of mathematics has also been used by Viskic and Petocz (2006). They found that at least some of the students experienced a growth in awareness, gained through the written reflections.

For adults returning to mathematics after some years away it is often the case that the previous education did not lead to that the student reaching his or her goals, leaving them without formal qualification for further studies. It has been shown that for adults returning to mathematics, the motivations are often focused on the formal qualifications and not on learning mathematics as a subject (Strässer & Zevenbergen, 1996). But there are other rationales for studying mathematics, as described by Swain, Baker, Holder, Newmarch and Coben (2005): “The main triggers are to prove that they can succeed in a subject where they have previously experienced failure; to help their children; for understanding, engagement and enjoyment; and to get a qualification for further study.” (p.86). Similar motivational factors for adults attending mathematics courses in folk high school in Finland were found by Hassi et al., (2010).

Another thread in research on formal mathematics education of adults concerns adults’ building of conceptual understanding. This issue has been investigated in relation to different types of mathematical content, e.g., functions (Lane, 2011), proportional reasoning (Díez-Palomar, Rodríguez, & Wehrle, 2006), probability (Khazanov, & Prado, 2010), fractions (Baker, Czarnocha, Dias, Doyle, & Kennis, 2012) and rational numbers (Doyle, Dias, Kennis, Czarnocha, & Baker, 2016). Lane (2011) highlighted the benefits with visual imagery to enhance a college algebra student’s understanding of the concept of function in a case study. Also, in a case study of six female adults, Díez-Palomar et al., (2006) conclude that the students had difficulties with the characteristics of the linear function (c.f. Karsenty, 2002). In a study by Khazanov and Prado (2010), misconceptions about probability were addressed by teaching activities aiming to trigger cognitive conflicts thereby leading students to build new understandings of the concepts. The results suggest that it is possible to develop students’ conceptual understanding of probability and correct their misconceptions by targeting the misconceptions directly.

The studies above are all small-scale qualitative teaching experiments. An example of a quantitative study is Baker et al., (2012) which builds on the assumption that fractions are the most difficult topic for students in community college. To enhance students’ knowledge of operator and measure, Baker et al., used the Kieren-Behr’s model for sub-constructs where the part-whole concept is described by the sub-constructs: ratio, operator, quotient and measure (Behr, Lesh, Post, & Silver, 1983). The authors conclude that flexibility in cognitive pathways, as suggested by Grey and Tall (2001), is beneficial for adults’ growth in conceptual understanding of operator and measure. In the second part of this study the authors used transcripts from students’ work with problem solving involving fractions and rational numbers (Doyle et al., 2016). The results show that the concepts of part/whole-circles and number line-measure represented in visual form acted as a catalyst for students’ reasoning.

A general difficulty in formal education of adults concerns prior knowledge. Mathematical skills and knowledge vary greatly between adults enrolling in the same course. If teachers assume that the learners possess the

required prior knowledge some students may lose their faith in coping with the course due to expectations they cannot fulfill. Using the lens of Brousseau's (1997) didactical contract, Gill and O'Donoghue (2007) investigated how a service mathematics university course was perceived, planned, delivered, evaluated, assessed and experienced. The results show that there is a mismatch between teachers' and learners' expectations of what the learners' prior knowledge is. This leads to students' negative affective feelings about mathematics as a subject being likely to increase and that the likeliness to drop out increases.

A problem related to such an expectancy mismatch is that some of the difficulties of adult students starting a new course may be related to fundamental mathematical concepts that for a very long time have constituted learning obstacles for the students. In an approach similar to master learning, McDonald (2013) tried to remedy such problems by using a step-by-step teaching design. "In SBST, [step-by-step-teaching, my remark] information is explored in a step-by-step manner so that the learner has to show understanding of previous information before moving on" (McDonald, 2013, p. 359). Step-by-step teaching builds on a researcher-designed process aiming to take the adult learner, step-by-step, from her present level of understanding to the required level. A sample of 35 students participated and was compared with a control group. Besides the increased conceptual understanding of fundamental concepts that previously had constituted learning obstacles for several years, students' mathematical achievement, attitudes, beliefs, and self-confidence toward mathematics learning also improved.

In summary, a lot of the research on adults learning mathematics in formal settings has identified specific issues for adult education, but there is still much work to be done when it comes to developing methods and principles for teaching. In a review and summary of research on adult mathematics education in North America (1980-2000), made by Safford-Ramus (2001), she concludes that:

The body of doctoral research in adult mathematics education is small but cohesive. Much is known about the symptoms of student problems and work now needs to be continued or begun to devise and test "treatment plans" to help adult mathematics students gain confidence and to become successful in their studies of mathematics at all levels of the education system. Learning theories and teaching methodologies from traditional system research need to be analyzed and adapted for adult populations and then tested via doctoral studies. (p. 5)

While following Safford-Remus' (2001) advice that learning theories need to be analyzed and adapted for adult populations it is important to draw on what is already known from existing research. From the review presented above we can see that weak motivation and negative affective feelings towards mathe-

matics are more common among adult learners than among children and adolescent learners. The understanding of basic concepts such as fractions, proportions and functions is a challenge for many adults, which is also mirrored in a large variety in prior knowledge and mathematical skills for adults taking up education. Furthermore, the issue of transfer from non-formal and informal mathematics learning to formal learning as well as the other way around is of interest when designing adult education. Drawing on these insights, individualized instruction of adults in the Swedish prison education program may be informed and organized so that the students are given opportunities to succeed with their studies and reach their individual goals.

Moving away from the field of adults' mathematics learning, the next section gives a brief overview on motivational theories in general. The purpose is to give a backdrop for a theoretical consideration (see section 3.4) regarding the conceptual framework adopted for the study presented in paper 2.

## 2.3 Motivation

What makes people do stuff that requires significant emotional, physical and cognitive engagement? What is it that makes different people in seemingly similar situations engage in different ways and in different levels in a task? These are questions that have engaged researchers in motivational theory for decades. To access the development and the state of affairs in the large field of research on motivation theory, I refer to review papers from different time points (Graham & Wiener, 1996; Middleton, & Spanias, 1999; Schukajlow, Rakoczy & Pekrun, 2017) as well as other well-cited papers in the field.

The dominant researchers in the dawn of the research field thought that human motivation was too complex to study directly. Retrospectively we can understand that this view was tightly connected to the research methods used at the time, namely to deprive an organism of something necessary for survival and study the reactions (Graham & Wiener, 1996). This method was in turn tightly related to the dominant theory of the time, the drive theory, or drive reduction theory, of Hull (1943) and Spence (1958). Here physiological needs or secondary drives, like the human need for money, were thought to stimulate activity and make organisms leave their resting state. Needless to say, while this early motivation theory made great claims of generality, the empirical grounding in rather simple experiments on rats and other animals made the jump to understanding the motivation of humans very large (Graham & Wiener, 1996). Perhaps more importantly, drive theory did not explain why humans and other animals sometimes used great energy for efforts that seemingly did nothing to reduce their basic needs. Later, the research of motivation shifted to a more cognitive approach and the study of motivation moved towards choice and persistence.

Quite early on, motivation became a major focus in educational psychology (Graham & Wiener, 1996). Since education plays a fundamental role in society and engages a large portion of the population. Since the seventies, it is therefore in the field of educational psychology that a major share of the motivation research has been carried out. The mechanistic perspective described above proved to be inadequate for explaining educational phenomena. As of yet, there is no great unifying theory of motivation, instead competing or overlapping theories are used to explain different phenomena. Several of these theories try to explain how the student's relation to the educational environment affects the student's level of motivation, typically by modeling the mechanism through some psychological construct. Some such theories are briefly described below.

In self-efficacy theory (i.e. Bandura, 1997), the individual's beliefs concerning how successful she might be in handling the task is in focus. The assumption is that if an individual has high levels of self-efficacy in relation to some task, she will exert more effort and show greater perseverance and determination when working with the task. In comparison an individual with lower self-efficacy in relation to the task might instead give up (Stajkovic & Luthans, 1998). It has been demonstrated by several researchers that self-efficacy beliefs predict mathematics performance across a wide range of measures (Bandura, 1977; Pajares, 1996). Self-efficacy theory has been employed quite extensively in general education but was largely ignored in mathematics education research for many years (Schukajlow et al., 2017). This has changed and there has been research conducted to try to affect efficacy through interventions as well as research on development of instruments for measuring self-efficacy specifically for the field of mathematics (Street, Malmberg & Stylianides, 2017).

A theory that has some resemblance with self-efficacy theory is expectancy-value theory (Wigfield & Eccles, 2000). The resemblance concerns both research methods and the aim of trying to explain levels of motivation and its effect on achievement. But, while self-efficacy theory focuses on the individuals' beliefs about her ability to handle a task, expectancy value theory adds the component of the individual's beliefs of how important the task is. In essence, expectancy value theory models engagement in a subject by means of expectations of success and the subjective value of the task. Using this two-factor construct, expectations and value, it has been shown that expectations of success is a stronger predictor of academic achievement while beliefs about the value of a subject is a stronger predictor of engagement and educational choices (Eccles & Wigfield, 2002; Wigfield & Eccles, 2000). A similar theory, attribution theory, deals with motivational effects of what the individual attribute as the causes of success or failure (Weiner, 2000).

Common to these theories is that they all deal with individuals' beliefs or appraisals of a task situation and from that try to model or measure engagement and achievement. A different type of approach that has been used in

mathematics education is self-determination theory (Deci & Ryan, 1985; Ryan & Deci, 2000). A basic tenet of this theory is that motivation comes in different qualities and not only in different quantities. In particular, self-determination theory separates intrinsic motivation from extrinsic motivation. A task or activity is intrinsically motivating when completing or carrying out the activity is motivating in itself. An activity is extrinsically motivating if it leads to some outcome that is separable from the activity itself. Through empirical work, Deci and Ryan (1985) noted that intrinsic motivation typically led to better performance than extrinsic motivation. For example, across ages from elementary school to college level it was shown that students that reported greater levels of intrinsic motivation displayed greater levels of conceptual understanding as well as better memory retention than other students (Benware & Deci, 1984). Therefore, conditions for stimulating intrinsic motivation have been widely studied and it has for example been found that innate needs for autonomy, competence and relatedness are driving forces of intrinsic motivation (Ryan & Deci, 2000). In early research, intrinsic and extrinsic motivation seemed to be mutually exclusive. Later this condition was relaxed and it was found that extrinsic motivation could be of different kinds in itself, corresponding to different levels of internalization and different combinations of self-determined behavior versus externally controlled behavior (Deci, Valleraud, Pelletier & Ryan, 1991). Consequently, in self-determination theory the constructs have been updated and in current descriptions extrinsic motivation is described as a continuum ranging from externally regulated to integrated, where the latter is considered more desirable because of its association to better achievement (Ryan & Deci, 2000).

The research on motivation has to date helped us understand the structure of motivation as well as how different types of motivation might predict different types of learning on a group level. There are however very few intervention studies aiming to change the learning conditions to achieve more desirable types of motivation (Schukajlow et al., 2017). One reason might be that even if motivational research on group level can predict outcomes in engagement and achievement, it is not very easy to explain individuals' behaviors. Explaining individuals' behaviors is the research interest in the study presented in this thesis (paper 2).

One well-cited study dealing with motivation on individual level is William and Ivey's study from 2001. The middle-school student Bryan was investigated using a number of theories (Williams & Ivey, 2001). They all failed to explain Bryan's motivation. He exhibited radically different levels of motivation and engagement depending on the situation. Williams and Ivey applied theories on: causal attribution, self-efficacy, perceived usefulness, goal orientation, and volition. It turned out that each of these theories explained separate components of Bryan's behavior, but none of them provided a useful explanation of the behavior in its totality.

It can be said that, in their analysis, Williams and Ivey (2001) display a research gap concerning individuals' interest for mathematics as a subject. Motivation and engagement may not only depend on relationships between individuals and task situations that can be understood in terms of efficacy, attribution of success and other types of phenomena dealt with general motivational research. It may well be issues concerning beliefs of the nature of mathematical practice that affects people's motivation and engagement. This in particular, has implications for mathematics teachers working with trying to improve the motivation of students. Williams and Ivey conclude:

Efforts to help Bryan re-conceptualize mathematics must be built on the foundation of respect for Bryan's responsibility for his choices, and his feelings about self-expression. Such efforts need not focus on ideas, such as success and failure, which do not seem important to Bryan. (Williams & Ivey, 2001, p.97)

The position that it is the students that shall adapt to predetermined teaching can be found in a great deal of the motivational research (c.f. Stipek et al., 1998). For example, in their review of the field, Graham and Weiner (1996) state that the main question of motivational research has been how to get children to "accept the basic premise that learning, schooling, and mastery of the material that adults prescribe are important?" (p. 81). For large scale instruction this is indeed an important premise because it is not possible to cater to the needs of every student individually. But, despite all efforts made in the motivational field I could not see how any of the above-mentioned theories could provide a coherent applicable theoretical lens that explains and describes the motivation for individual adults to learn mathematics. The reason, I believe, lies in the epistemological approach that comes from how to answer the question: Shall students adapt to teaching or shall teaching adapt to students?

I believe the latter is a more promising approach for organizing individualized instruction for adult mathematics students. I therefore chose Mellin-Olsen's educational concepts as a conceptual framework for explaining adult students' rationales for studying mathematics in prison (to be further elaborated on in section 3.4).

In study 3, the timing of feedback is of central importance. Therefore, the next section gives a brief overview of what is known about the timing of feedback and about contradictory results on when to deliver feedback. Also, in this overview we have moved out from the field of adults' mathematics learning solely for the lack of studies on timing of feedback for adult mathematics learners.



## 2.4 Timing of Feedback

Gathering information to elicit students' learning is essential for providing feedback that moves the learners forward (Ginsburg, 2009; Wiliam, 2007). The information gathered serves two purposes. For the teacher, in a formative teaching practice, the information is used to plan effective instruction. According to Ginsburg (1981; 2009), there are three main methods for gathering information on students' learning; observations, tests and clinical interviews. Once the information has been gathered the teacher has to choose how to deliver feedback so that it can be meaningful for the student. Furthermore, in a formative teaching practice, inferences about student levels of knowledge are used to decide the next step of instruction. But, planning instruction on the basis of gathered information of students' learning has been found to be more difficult for teachers than to analyze and elicit students' knowledge (Heritage, Kim, Vendlinski, & Herman, 2008).

One issue when giving feedback is the timing. The question is: Shall feedback be given immediately after the students have delivered a solution to a task or a problem, or should it be delayed? The issue of timing has been a recurring theme in research on formative assessment, but the recommendation differs. Advocates of immediate feedback argue that errors need to be highlighted and refuted before they consolidate in students' minds (Phye & Andre, 1989). The antagonists, the supporters of delayed feedback, generally base their argument on the interference-perseveration hypothesis, as proposed by Kulhavy and Anderson (1972). The hypothesis asserts that initial errors do not compete with to-be-learned correct responses if corrective information is delayed. This is because errors are likely to be forgotten and thus cannot interfere with retention. The hypothesis was born from data on students' performance on a multiple-choice test with high school juniors and seniors on topics in introductory psychology under various conditions of immediate and delayed feedback.

In our judgement, the explanations advanced to this point fail to adequately account for why the DRE [Delay-retention effect, min anmärkning] occurs with meaningful material. Our explanation is very simple: learners forget their incorrect responses over the delay interval, and thus there is less interference with learning the correct answers from the feedback. The subjects who receive immediate feedback, on the other hand, suffer from proactive interference because of the incorrect responses to which they have committed themselves. This explanation will be called the interference-perseveration hypothesis. (Kulhavy & Andersson, 1972, p. 506)

According to Shute (2008) this hypothesis has been questioned by several researchers (e.g., Kippel, 1974; Newman, Williams, & Hiller, 1974; Phye & Bailer, 1970, in Shute, 2008). However, there is no conclusive answer regarding which one is better, delayed or immediate feedback. But, there is a wide

support for that immediate feedback works better than delayed for promoting procedural skills. Also, there is some support for that delayed feedback works better for building conceptual knowledge. Timing is also discussed in a review of Hattie and Timperley (2007). They conclude:

There has been much research on the timing of feedback, particularly contrasting immediate and delayed feedback. Most of this research has been accomplished without recognition of the various feedback levels. For example, immediate error correction during task acquisition (FT) can result in faster rates of acquisition, whereas immediate error correction during fluency building can detract from the learning of automaticity and the associated strategies of learning (FP). Similarly, in their meta-analysis of 53 studies, Kulik and Kulik (1988) reported that at the task level (i.e., testing situations), some delay is beneficial (0.36), but at the process level (i.e., engaging in processing classroom activities), immediate feedback is beneficial (0.28) (see also Bangert-Drowns et al., 1991; Brackbill, Blobitt, Davlin, & Wagner, 1963; Schroth & Lund, 1993; Sturges, 1972, 1978; Swindell & Walls, 1993). (Hattie & Timperley, 2007, p. 98)

In similar ways, Clariana, Wagner and Murphy, (2000) claim that immediate feedback is likely to be more powerful for procedural skills in task acquisition while delayed feedback are more powerful for more complex problems. The hypothesis put forward by the authors is that more difficult items needs to be processed on several levels while easy items do not and therefore they benefit from immediate feedback.

This overview of what is known about timing of feedback will be discussed in relation to the results presented in paper 3; a retrospective analysis of how a shift of the feedback situation affected a student's mathematical learning.

## 3 Theoretical Considerations

I believe that mathematics is an activity in which concepts are invented, learned and used. The complexity of mathematical conceptualization therefore originates in mathematics being both a body of knowledge and an activity. In my opinion mathematical knowledge has a communal side, where the epistemology is simple and transparent due to how mathematics is presented through written axioms, definitions and theorems. However, mathematics also has a psychological side, where the epistemology is obscure and hidden. Depending on what aspects of mathematics one is interested in, one's theories of conceptualization needs to be chosen accordingly.

In the following section I provide a personal reflection on the role of theories. Then I discuss theories for conceptualization and my considerations for choosing *the theory of conceptual fields* over other well-established theories. Thereafter I present the *theory of conceptual fields*, *the multiplicative conceptual field*, *scheme theory*, *the theory of representations*, and finally Mellin-Olsen's educational concepts *the S- and the I-rationale for learning*. All of these theories are crucial to my analysis of the studies presented in the attached papers.

### 3.1 The Role of Theories

In the present work I used a variety of theories and theoretical frameworks as tools for different purposes. My interpretation of how to put theory to work has evolved over time. At the very beginning of my graduate studies I attended a course at Umeå University where we discussed the difference between theory and theoretical frameworks. My naive interpretation at the time was that there is a clear distinction between theory and theoretical frameworks. As I understood it at the time, a theory always has explanatory power while a theoretical framework does not always explain phenomena. The role of the latter was rather to label different components and phenomena in a coherent matter. Today, after some years have passed, my understanding differs. While theoretical frameworks or theoretical constructs are structures that provide classifications and terms for useful concepts theoretical frameworks can also act as delimited theories with power to explain phenomena. It is just that theoretical

frameworks rarely make claims of providing explanations outside of the observed object (c.f. Sfard, 1991) like larger coherent theories with explanatory power does, for example Chevallard's Anthropological Theory of the Didactic (ATD) (Bosch, & Gascón, 2014) or Brousseau's (2006) Theory of didactical situations in mathematics (TDS).

Now I see no clear difference between theories, theoretical constructs and frameworks. A theoretical framework can be a theory, but it can also be parts of a theory combined with theoretical concepts gathered from parts of other theoretical constructs, c.f. bricolage (Lawler, 1985). What is scientifically useful and sound depends solely on the basis of the type of research you are conducting. Theory-driven research aims to underpin, test, or expand an already existing theory, while problem-driven research departs from problems in practice (Arcavi, 2000). In addition to the already mentioned drivers for research, Schoenfeld (1992) adds method-driven research, where methods are tested in, for example, new settings or other fields of research. Research can also be data-driven (Jankvist, 2009). Typically, researchers then use large sets of data, like TIMMS (Trends in International Mathematics and Science Study), and see what they can find.

Theory may serve six different purposes in research according to Niss (2007a, 2007b). First, theory may provide an explanation of some observed phenomenon. Second, a theory may provide predictions so you can hypothesize the effects of particular causes. Third, theory may provide guidance for action and behavior in order to achieve certain goals. Fourth, theory may serve as a safeguard against unscientific approaches, enabling us to avoid inconsistent choices with regards to the research process of asking questions, collect data and, analyze and interpret data. Fifth, theory may provide protection against attacks from the outside since a solid theoretical foundation may strengthen the results of the research so it can be scrutinized from within and outside the field of research. Finally, theory may provide a structured set of lenses through which phenomena may be investigated.

The role of frameworks as structures for conceptualizing and designing mathematics education research studies was elaborated on by Lester (2005). He discusses the educational anthropologist Margaret Eisenhart's (1991) three identified frameworks: theoretical, practical, and conceptual, as well as their characteristics. For the theoretical frameworks the theory itself works as a special kind of framework, where the researcher uses accepted conventions of argumentation and experimentation associated with the theory. The gathered data in the research supports, extend, or modifies the theory that serves as a backdrop for the framework in question. A practical framework, on the other hand, is based on accumulated practice knowledge on what actually has been proven to work. Practical frameworks address problems for the people directly involved, which could be a strength in comparison to theoretical frameworks. However, a drawback with practical frameworks are that they are only locally

generalizable. But, if the goal of the research is to say something about a particular context, at the expense of the overall generalizability of the results, this may be a strength instead of a weakness. If the researcher instead seeks to justify generalizations they should rely on conceptual frameworks that are based on previous research and theory. In contrast to theoretical frameworks sprung from the base of one theory, conceptual frameworks can build on a variety of sources and can be “based on different theories and various aspects of practitioner knowledge, depending on what the researcher can argue will be relevant and important to address about a research problem” (Lester, 2005, p. 460).

Jankvist (2009) compared Schoenfeld’s (1992) categorizations, method-driven, theory-driven or problem-driven research with Lester’s interpretation of the theoretical, practical, and conceptual frameworks. He interpreted that theory-driven research corresponds to theoretical frameworks, whereas method-driven research corresponds to practical frameworks. The problem-driven research can correspond to both conceptual frameworks and practical frameworks, depending on the goals of the investigations.

The studies in this thesis all sprung from problems in practice. In that sense they are all problem-driven. However, to approach the problems I have used both theory-driven and method-driven questions. Thus, I have used theory both to provide explanations of some observed phenomena and to provide predictions of certain phenomena. But most importantly, theory has given a structured set of lenses through which my observed phenomena have been approached, observed, studied, analyzed and interpreted.

## 3.2 Theories of Mathematical Conceptualization

There are several educational theories aiming to model the development of conceptual knowledge. Examples of well-established theories are Tall and Vinner’s (1981), drawing on Vinner and Hershkowitz’s (1980) idea of concept image and concept definition, Sfard’s theoretization on reification (1991), and APOS theory (e.g., Asiala et al., 1996; Dubinsky & McDonald, 2002). For Tall and Vinner, progress in conceptualization means that someone’s images of a concept evolves in the direction of the formal mathematical definition. However, many mathematical concepts (e.g., number, addition) are taught without providing formal definitions, so the metaphor is of limited scope. Both Sfard and APOS theory model concept formation as a linear process. In Sfard’s case through condensation - interiorization - reification, and in APOS theory through action - process - object - schema. Of course, neither Sfard nor Asiala et al., actually claims that concept formation is a strictly linear process. Sfard problematizes her own model of conceptual development with the thesis of the vicious circle of reification: “*the lower-level reification and the higher-level interiorization are prerequisite for each other!*” (Sfard, 1991, p. 31) Asiala et

al., (1996) clarify their stance about modelling concept formation: “It is important to emphasize that although our theoretical analysis of a mathematical concept results in models of the mental constructions that an individual might make in order to understand the concept, we are in no way suggesting that this analysis is an accurate description of the constructions actually made.” (p. 20) Science is reduction and linear models are reductions of reality that serve well for different purposes in research. However, if you instead conceptualize concept formation from the perspective that concepts develop in harmony with other concepts that are prerequisites for each other's development, another simplification of reality is needed.

From this point of departure (c.f. Brandom, 2009; Vergnaud, 2009), the essence of the insight that concepts are always evolving in relation to other concepts has been nicely expressed by the philosopher of language, Robert Brandom: “Cognitively, grasp of just one concept is the sound of one hand clapping” (2009, p. 49). Brandom’s view that concepts cannot exist in isolation is shared by the French psychologist Gérard Vergnaud. For that reason, he has developed the *Theory of conceptual fields* as a structured set of lenses to provide an explanation of why mathematically simple concepts are psychologically complex. In the study presented in paper 1, data is analyzed with Vergnaud’s theories of conceptual fields and representations. All situations that can be analyzed as simple and multiple proportion problems constitute the multiplicative conceptual field (MCF), where concept knowledge grows in a context of other concepts, situations and representations (Vergnaud, 1988). In paper 4, I used Vergnaud’s theoretical constructs to analyze a particular student’s solution of one of the items from the test presented in paper 1. Here I use Vergnaud’s (1998a) theory of representation, where he theorizes that all higher thinking is mediated by systems of signs, in line with Vygotsky (1962). In the next section I describe the work from Vergnaud that I use in paper 1 and 4. First, the theory of conceptual fields (Vergnaud, 2009), and the multiplicative conceptual field (Vergnaud, 1988, 1994, 1998b). Second, I describe scheme theory (e.g., Piaget, 1970; Vergnaud, 1998b; von Glasersfeld, 1989), which is a central component in the theory of conceptual fields and third, the comprehensive theory of representations (Vergnaud, 1998a).

### 3.3 The Theory of Conceptual Fields

Vergnaud (2009) claims that a concept has no meaning without the presence and meaning of other concepts. To illustrate this claim you can consider for example the equal sign. Without the presence of expressions on the right- and left-hand side of the sign, it cannot be filled with meaning. The insight that a concept cannot exist in isolation of other concepts is why Vergnaud (e.g., 1997, 1998b, 2009) developed: *The theory of conceptual fields*.

The theory of conceptual fields is a developmental theory. It has two aims: (1) to describe and analyse the progressive complexity, on a long- and medium-term basis, of the mathematical competences that students develop inside and outside school, and (2) to establish better connections between the operational form of knowledge, which consists in action in the physical and social world, and the predicative form of knowledge, which consists in the linguistic and symbolic expressions of this knowledge. (Vergnaud, 2009, p.83)

A conceptual field consists of a set of different concepts tied together and a set of different situations where the concepts apply. According to Vergnaud (c.f. 1983, 1988, 1994) a variety of situations are necessary to give a concept meaning. Conversely, a class of situations cannot be analyzed with one concept alone. Rather, several related concepts are required to understand any situation. Conceptual fields consist of such clusters of situations and concepts.

The learning of different properties of the same concept develops over several years (Vergnaud, 1997). Everyone with experience from teaching and learning mathematics knows that a mathematical definition is not enough to extract properties of a concept. Hence, if you want to analyze how mathematical concepts develop in individuals' minds, concepts need to be considered from a psychological perspective. Vergnaud (1997, 1998b) presents a concept,  $C$ , as a triple of three sets,  $C = (S, I, R)$ .

S: the set of situations that make the concept useful and meaningful.

I: the set of operational invariants that can be used by individuals to deal with these situations

R: the set of symbolic representations, linguistic, graphic or gestural that can be used to represent invariants, situations and procedures. (Vergnaud, 1997, p. 6)

S: The set of situations that make the concept useful and meaningful refers to the different situations where knowledge of the properties of the concept is necessary for dealing with the situation. For some concepts, like addition, the number of applicable situations is limited. However, the conceptual field of multiplicative structures is far more complex and elusive to describe. It consists of all situations that can be analyzed as simple or multiple proportion problems (Vergnaud, 1988). As mentioned above, the number of situations where proportional reasoning applies dominates the content taught in compulsory school mathematics (c.f. Behr Harel, Post, & Lesh, 1992; Hilton, Hilton, Dole, & Goos, 2013; Karplus, Pulos, & Stage, 1983; Lamon, 2007; Sowder et al., 1998). The mastery of these situations requires a number of concepts such as linear and  $n$ -linear functions, vector space, dimensional analysis, fraction, ratio, rate, rational number and multiplication and the inverse operation division. Each one of these concepts in themselves applies to a range of situations.

I: The set of operational invariants that can be used by individuals to deal with these situations are concepts-in-action and theorems-in-action. The operational invariants can be used by individuals to identify and select relevant

information to deal with these situations (see for example Vergnaud 1983, 1988, 1994, 1997, 2009). An operational invariant is some organized way of dealing with situations that remain constant (invariant) for all situations in a class. Note that one can neither define what constitutes such a class before knowing of the invariants, nor speak of the invariants before knowing the class.

Theorems-in-action are the mathematical relationships that students take into account to deal with a situation. They constitute the operation, or the line of operations, chosen to solve the problem at hand. These theorems can be more or less explicit to the student. To highlight that students often operate with implicit theorems that they cannot express verbally, Vergnaud defined this invariant as theorems-in-action, and not just theorems. A theorem in action can be true or false, because it is a proposition. This is an important distinction that separates theorems-in-action from the other operational invariant, concepts-in-action (Vergnaud, 2009). Concepts-in-action cannot be true or false, only relevant or irrelevant to the situation. The suffix “in-action” signifies that the concepts in play are several for each situation and often implicit. That is, if you ask students what concepts they have used in their solution it is unlikely that they can account verbally for the meaning of each of them.

R: The set of symbolic representations, linguistic, graphic or gestural, that can be used to represent invariants, situations and procedures refers to all semiotics in play in and between individuals. The role of representations for concept formation is to make knowledge explicit and communicable. An operational invariant can be recognized and used with more ease and efficiency when it is associated with a particular word, gesture or other symbol. Explicit knowledge can be shared with others by means of communication while implicit knowledge cannot. (Vergnaud, 1998a). Concepts- and theorems-in-action are usually limited to small classes of situations and are hard to connect to each other, but explicit concepts are always tied together in systems, which is what Vygotsky (1962) calls scientific concepts. For the progressive development from implicit concepts- and theorems-in-action to scientific concepts, language plays a crucial role (Vergnaud 1998a). This was recognized by Vygotsky (1962) as well as by Piaget (1970). That is why Vergnaud (2009) developed: *The theory of conceptual fields* on the legacy of both Piaget’s and Vygotsky’s work.

In the next section I present the multiplicative conceptual field (e.g., Vergnaud 1988), that is of importance for the analysis of the items in paper 1 and 4.

### 3.3.1 The Multiplicative Conceptual Field

The multiplicative conceptual field is defined by Vergnaud (1988) as “all situations that can be analysed as simple and multiple proportion problems and



for which one usually needs to multiply or divide” (p. 141), i.e., all situations that can be analyzed with proportional reasoning.

A proportion is defined as a statement of equality of two ratios  $a/b = c/d$ . Proportion can also be defined as a function with the isomorphic properties  $f(x+y) = f(x) + f(y)$  and  $f(ax) = af(x)$  (Vergnaud, 2009). A function,  $A(x,y)$ , can also be linear with respect to several variables, ( $n$ -linear) functions. For example the area functions for a rectangle with sides  $x$  and  $y$  is bilinear (2-linear) since  $A(x,y) = xy$  and it is easy to check that this function is linear with respect to each of its variables when the other is considered constant. (Ahl, 2019, p. 8)

As we can see from the definition we need three known quantities to be able to analyze the relations in a proportion. It may be close at hand to think of multiplication as involving three magnitudes, but this is usually not the case. Considering the multiplication 3 times 5 equals 15. The situation appears to have three present numbers. But multiplication always holds the implicit neutral element, 1, giving the expression 3 times 5 equals 15 times 1. In fact, even the simplest situation where multiplication applies involves four numbers (Vergnaud, 1997). Consider the situation “how much money do I need to buy 5 cakes at 4 francs each?” from Vergnaud (1997, p. 21).

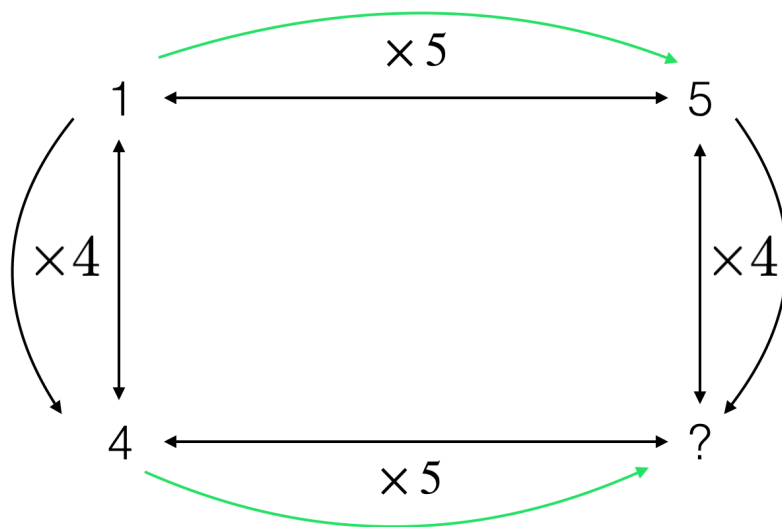


Figure 1. Multiplicative relations for cakes and francs. The curved green arrows represent reasoning within a measure space, either francs or cakes. The curved black arrows represent reasoning between measure spaces.

The implicit ‘1’ derived from the fact that 1 cake cost 4 francs makes the situation a quaternary multiplicative relationship that can be analyzed in two different ways. Using the scalar operator 5, since we want to find the cost for 5 times as many cakes as 1, will give the, within measure space, reasoning 5 times 4 (francs) equals 20 (francs). Within measure space refers to transformations within the same measure, here francs. The theorem in action is  $P(5) = 5P(1)$ , where  $P$  is the price and the scalar 5 is a pure number without dimension. This theorem builds on the additive property of the linear function:

$$P(5) = P(1 + 1 + 1 + 1 + 1) = P(1) + P(1) + P(1) + P(1) + P(1).$$

Using the function operator from the two different measure spaces, cakes and francs, will give the reasoning 4 (francs/cake) times 5 (cakes) equals 20 (francs). Here we might have a learning obstacle, as it is not easy to see how multiplying a rate with cakes gives francs (Vergnaud, 1997). This interpretation requires either some dimensional analysis or an instrumental understanding that the ‘way over one’ usually is an efficient scheme for these situations (to be further explained in chapter 5).

In line with the above presented constructs from Vergnaud, I analyzed students’ actions, as observed by the test and the clinical interview, in relation to the operational invariants in play to elicit students’ prior knowledge on proportional reasoning (paper 1).

### 3.3.2 Scheme Theory

To understand the relationship between cognition and empirical objects in the world, the concept of scheme was first introduced by Kant. The question in focus was the nature of knowledge. The empiricist tradition of Locke and Hume saw knowledge as originating from impressions in our senses. In contrast, in the rationalist tradition Descartes and Leibniz saw knowledge as originating from the inner mental activity of reason. Kant theorized on how the scheme concept can combine these two traditions.

Later on, to give a response to the, at the time, dominant behaviorist tradition, Piaget elaborated on Kant’s scheme theory. Piaget claimed that mental constructs shape our interpretation of the world and our actions. (Inhelder & Piaget, 1958; Sriraman & Nardi, 2012; Vergnaud, 1996). But these constructs are far from arbitrary, as they also need to allow the person to act in the world. Piaget transferred the fundamental idea of adaptational phylogenesis in the biological sense to adaptational ontogenesis in the cognitive sense. This transposition enabled Piaget to draw the conclusion that knowledge is a result of interaction between experience, prior knowledge, and action on environment

(Piaget, 1971). At the same time, mental constructs are shaped by the experiences and actions in a dynamic equilibrium process. The scheme is the construct that Piaget (1970) gives the role of a mediator in this process (Cobb & von Glasersfeld, 1984). Schemes hence have to be rigid enough to organize behavior associated to classes of known situations. But they also have to be flexible enough to allow association to new situations where the behavior is still viable, and to be adaptable enough to allow re-organization when new situations so require. This is what Piaget described in his equilibrium theory where assimilation and accommodation are central constructs to describe how humans gain knowledge. All assimilation and accommodation are made in relation to schemes (Piaget, 1986; von Glasersfeld, 1989). Piaget defines schemes as “Whatever is repeatable and generalizable in an action is what I have called a scheme, and I maintain that there is a logic of schemes.” (Piaget, 1970, p.42).

Piaget described three characteristics of schemes: behavioral, symbolic, and operational schemes (Piaget, 1952; Piaget, Grize, Szeminska, & Bang, 1977). The organized patterns of *behavior schemes* are used to represent and respond to experiences and object in the world. The inner mental *symbolic schemes* are used to represent aspects of experience. The internal mental activities in the form of *operational schemes* are used on objects of thought. In spite of the fact that much of Piaget’s work hinges on schemes, he did not explicitly define scheme in his writing. But based on Piaget’s work, von Glasersfeld (1989) interpreted and defined schemes as consisting of three parts: “(a) the child’s recognition of an experiential situation as one that has been experienced before; (b) the specific activity the child has come to associate with the situation; and (c) the result that the child has come to expect of the activity in a given situation.” (p. 219). Later on, when Vergnaud defines a scheme as *the invariant organization of behavior for a certain class of situations* (i.e., 1997, p.12) he does not keep the division between behavior, symbolic and operational schemes.

When analyzing mathematical intuitions, Vergnaud notes that the “analysis must be made in mathematical terms, as there is no way to reduce mathematical knowledge to any other conceptual framework” (Vergnaud, 1998a, p.167). This makes it appealing to think about student’s knowledge in terms of conceptual understanding. However, the concept “understanding” is both very difficult to use analytically for researchers and the mind does not seem to be organized by understanding, nor do behavior seem dependent on understanding (Vergnaud, personal communication, April 14th, 2017). Schemes, on the other hand, provide a theoretical connection between classes of situations in which we expect the students to act, and students’ actions. In particular, students’ actions can be explained in terms of adaptations of schemes applicable in the situation that students operate on. If one accepts that schemes organize behavior, you can create hypotheses about students’ actual schemes, by analyzing their observable behaviors. This is the reason why schemes are

a both an analytically and didactically useful theory for analyzing students' knowledge and why schemes are central in empirically grounded methodological analyses of epistemology of mathematical concepts. Empirical use of scheme theory hinges on that you can make predictions of people's ideas (schemes) by observing invariance in behavior across related situations.

A central idea in Vergnaud's (1996, 1997, 1998a, 1998b, 2009) definition of schemes (the invariant organization of behavior (or action) for a certain class of situations) is signified by the word invariant. Invariants are crucial for the analytical power of schemes in mathematics education. As described above, these invariants are defined in terms of (1) *theorems-in-action*, i.e., propositions explicitly or implicitly held to be true, and (2) *concepts-in-action*, i.e., predicative or categories held to be relevant in the class of situations associated to a particular scheme. A theorem-in-action is always either true or false. A concept-in-action can be relevant or not relevant for the situation.

Vergnaud's theorization this far largely builds on Piagetian ideas. But, in Vergnaud's view, Piaget's theory lacked many central components to be applicable to educational research (Vergnaud, 1996). Vergnaud rejects the view that mathematics is a language (1997). At the same time, he acknowledges the immense role that language plays for the mathematical domain of knowledge.

Mathematics as a science would not exist if there were no schemes and algorithms to make it functional in action. It would also not exist if there were no words or theorems to make it shared and debatable textual knowledge (Vergnaud, 1997). According to Vergnaud, Piaget did not pay enough attention to the importance of language in contrast to Vygotsky. The systematic treatment of how the use of signs and language organizes the mind is a major contribution to our understanding of knowing and learning from Vygotsky. Vergnaud combined Vygotskian insights with semiotics and developed a theory for the role of representation in scheme theory (Vergnaud, 1998a). In paper 4 I use Vergnaud's theory of representation to analyze a student's action.

### 3.3.3 The Theory of Representations

Vergnaud's work builds on the legacy of both Piaget and Vygotsky. That is why Vergnaud, in *A comprehensive theory of representations* (1998a), introduced an elaboration of scheme theory with extra attention to semiotics, inspired by Vygotsky. In line with many other researchers, Vergnaud saw the necessity of combining constructivist thoughts with socio cultural thoughts: c.f. social interactionism, (Bauersfeld, 1980; Voigt, 1985, 1989); socio constructivism, (Cobb & Yackel, 1996); the Italian school of mathematics education, (Bartolini Bussi, 1991); and the French theory (Brousseau, 1997), to account for the importance of language to mediate between cognition and empirical objects.

Vygotsky and Piaget were both concerned with how the mind mediates between the individual experience and the external world. They both shared a

passion for understanding and explaining how humans construct and exchange theories (Piaget, 1962). Yet, many will argue that Piagetian and Vygotskian ideas are not compatible, since in their respective pursuits to understand the mind, Piaget and Vygotsky differed in their approaches in their investigations. Piaget emphasized the role of the inner logical processes (Piaget, 1970), while Vygotsky on the other hand, emphasized the shaping role of the surrounding culture as formed by other humans (Vygotsky, 1962). Although Piaget and Vygotsky had different views of the genesis of knowledge, they were unified in their interest for the idea of interiorization, respectively in their works *Genetic epistemology* (Piaget, 1970) and *Thought and language* (Vygotsky, 1962). Interiorization is the process of making thoughts and knowledge a part of one's own being.

When mathematics learning is viewed from the perspective of the individual learner, it is worthwhile to try to analyze his or hers interiorization process. It is often argued that Piagetian based theory does not appropriately account for the constitutive role that culture has for cognition, by not offering a sufficiently articulated description of the role of human interaction (Radford, 2011). While this may be true, from the opposite point of view it can also be said that Vygotskian based theory does not offer a sufficiently detailed account of individuals' formation of specific concepts. The difference between the Piagetian and Vygotskian based approaches can be explained by referring to Bruner, according to whom humans make sense of the world in two principally different ways. By proof and logic, with the goal of explaining and finding truth, or by interpretation and narration, with the goal of understanding and producing plausibility. Bruner claims that: "Piaget was principally (though not entirely) preoccupied with the ontogenesis of causal explanation and its logical and empirical justification. [...] Vygotsky, on the other hand, was principally (though not entirely) concerned with the ontogenesis of interpretation and understanding" (Bruner, 1997, p. 72). These are two genres of explanation, but both "view the learning of mathematics as a process of internalization" (Cobb, 1989, p. 40). As Cobb (1989) discusses though: even if the intrapsychologically grounded theory of Vygotsky also deals with the phenomena of interiorization, there is far from as much detail provided on this process as in the interpsychologically grounded theory of Piaget. Indeed, Cobb and colleagues were themselves inspired by Vygotskian thinking when embarking on the route that later led to the extension of constructivism into socio-constructivism (Cobb, 1994). Yet, they chose to base their analysis of the social aspect of mathematical activity on symbolic interactionism instead (Voigt, 1995; Yackel & Cobb, 1996). As put by Cobb, "In my view, sociocultural theory has thus far been of limited usefulness in mathematics education research when actually formulating and improving instructional designs for supporting students' mathematical learning." (Cobb, 2006, s. 148)".

To summarize, while Piagetian and Vygotskian approaches have fundamental differences, there are also important points of overlap and their approaches can be seen as complementary. Since the core point of interest in the research presented here is the individual's conceptualization, I have chosen the theoretical approach of Vergnaud, who builds on Piagetian ground, complemented with a Vygotskian thought as, in the end, the role of language for concept formation and use cannot be circumvented (paper 4). Of special interest is the theoretical basis for the scheme concept and Vergnaud's theory of representations (1998a). This theoretization is used for analyzing a student's work in the study presented in paper 4.

In paper 2, I investigate how students' motivation can be characterized. In the literature review (section 2.3) I have argued that classical motivational theories mainly focus on how to get the learner to adapt to teaching. With individualized instruction it is the teaching that shall adapt to the students. Therefore, I have chosen Mellin-Olsen's *rationales for learning* as a conceptual framework for explaining adult students' rationales for studying mathematics in prison. I shall now move on to present this framework.

### 3.4 Rationales for Learning

Mellin-Olsen spent his life searching for answers to why so many intelligent students do not learn mathematics (Mellin-Olsen, 1987). Why do they stop learning in ordinary school and why do they fail to learn in remedial programs? For Mellin-Olsen, the ideology followed by those in charge of remedial programs imply that it is the pupil, rather than the curriculum that they have been exposed to, that has to be cured. This made no sense to Mellin-Olsen (1987). From a diametrically opposite position, he advocated that if a student stops learning it should be the teaching that should adapt to the student, instead of the other way around.

Mellin-Olsen was truly, deeply interested in the mechanisms driving students' actions in the mathematics classroom (1987). He searched to obtain thinking-tools, which could help to explain behavior. The ultimate goal was to build a theoretical basis which invites as many students as possible to mathematical knowledge. His research in the mathematics classroom led to the development of the educational meta-construct instrumentalism. This should not be confused with instrumental understanding (Skemp, 1976) although instrumentalism can lead to either instrumental or relational understanding (Mellin-Olsen, 1981).

Instrumentalism is defined by Mellin-Olsen as a rationale for learning. He based his argumentation on the fact that situations in school may occur where the learner is unable to place the learning situation in any other context than the instrumental one of school learning. The sole meaning related to the situation is then the fact that the situation belongs to the school context, and school

is a place where it pays to learn. To understand students' behavior in these situations Mellin-Olsen concluded that Activity theory was not enough (for more information see Mellin-Olsen, 1987). In Activity theory, society and the individual are studied dialectically. But individuals' actions belong to the individual, not the observer. That is, Activity theory cannot explain how to interpret the behavior of the students in order to understand their activities. For this understanding, Mellin-Olsen adapted complementary concepts from Mead and Bateson to theorize on students' rationales for learning in a school context.

The social behaviorist Mead's (1934, 1965) concept of the generalized other (GO) plays a crucial role in how Mellin-Olsen theorized on students' individual behavior and rationales for learning. The generalized other is the common attitudes and expectations (c.f. norms, my remark) to which the individual reacts in the social environment. The generalized other functions as the individual's referent for her behavior. The individual responds to how the generalized other reacts on her behavior. Thus, the individual experiences herself through the mirror of the generalized other. School is one of several generalized others, like family and all other communities (c.f. church, social media groups, sports clubs, my remark) the individual belongs to. Each carrying a unique set of common attitudes and expectations. To Mead, the individual's self is constituted by the I and the Me, where the I is spontaneous and unorganized in contrast to the Me that is controlling and organized. As a consequence, the Me is the social self; the individual's representation of society through the attitudes, expectations and meanings of the group. The I am a response to the social self, the Me. To Mead, the individual's self is a balance of the I and the Me. Mead's self has similarities with the Vygotskian conceptualization of the self, only Vygotsky's self is totally social (Mellin-Olsen, 1987).

The different generalized others exercise different control system over an individual's behavior (Mellin-Olsen, 1987). An individual's rationales for behavior is a dialectic concept. It belongs to the individual, while at the same time being a product of the individuals different generalized others. Drawing on Freud's conclusion that individuals' desires can be contradictory to the ethical and aesthetic demands of her personality, Mellin-Olsen expanded the generalized other to also be able to prohibit, oppress, favor and encourage behavior. Thus, it is not unusual for an individual to perceive contradicting generalized others. This leads to what Bateson (1973) calls a double-bind. Through generalization of Freudian theory into communication theory, Bateson gives the concept double-bind the power to examine the socialization process in education (Mellin-Olsen, 1987), where a double-bind arises when the individual is perceiving conflicting control systems from her different generalized others. The generalized other and the double-bind is of certain interest for understanding the educational meta concept instrumentalism as a rationale for learning.

Or, more precisely a rationale constituted by two different rationales for learning.

The two rationales for school learning that was identified by Mellin-Olsen (1981, 1987) was the social rationale and the instrumental rationale. The social rationale (S-rationale) says that the subject mathematics has an importance even beyond the walls of schooling. The importance is created through the individual's generalized others, her social setting that exists beyond school learning. The instrumental rationale (I-rationale), represent the reproduction of ideology, culture and the reproductive forces in society. While finding mathematics to be a meaningless waste of time, a student may very well be aware of the fact that mathematics is inevitable for moving on to the next level in the educational system. Mathematics is a gatekeeper for reproduction of labor force (Douglas & Attewell, 2017).

When the I- and S-rationale coincide, the learning situation is optimal (Mellin-Olsen, 1981). This is the case when the teaching practice provides relevant knowledge for the learners' social environment as well as supplies her with a mathematical experience that fulfills the desire to get good marks. Individualized instruction gives opportunities to design the teaching practice so that both the S- and the I-rationales can be evoked to, in the spirit of Mellin-Olsen, adapt the teaching practice to the student, instead of the other way around. It is from this point of departure that Mellin-Olsen's educational concepts are operationalized in paper 2.



## 4 Methodological Considerations

In this section I describe how to frame my research in terms of case study methodology. Further, I account for my ethical considerations regarding the double role of being both a teacher and a researcher and the considerations surrounding the data collection for the studies. Finally, I account for my part of the research in the co-authored articles in this thesis.

### 4.1 Case Study Methodology

The studies in this thesis are from a methodological perspective all conducted with a case studies approach. Case study is about the examination and analysis of a single phenomenon (Thomas, 2015). A case study is driven by a wish for understanding how and why something may have happened or why it might be the case. The definition adopted for the case study concept in this thesis follows Thomas (2011):

Case studies are analyses of persons, events, decisions, periods, projects, policies, institutions or other systems which are studied holistically by one or more methods. The case that is the subject of the inquiry will be an instance of a class of phenomena that provides an analytical frame—an object—within which the study is conducted and which the case illuminates and explicates. (p. 513)

The methodology of case study research builds on explaining something, the *object*, with something that has the potential to offer explanations, i.e. the analysis of the *subject* (Thomas, 2011). In other terminologies, when the case concerns a person that person is often called the subject. Sometimes an aspect of the case is also called the object of study. This terminology differs from the one used here.

Instead, in the typology of Thomas (2011) the subject is some historical unity framed as an instance of an analytically delineated phenomenon or class of phenomena. It is only through applying some theory or analytical frame that a class of phenomena can occur and the object is therefore the class of phenomena together with the analytical frame that delineates it. The subject is selected because it is interesting or telling about something, from which the object can be illuminated.

In paper 1, the possibility to elicit adults' various prior knowledge is investigated. By focusing on proportionality, a concept that is one of the most important connecting ideas in all of school mathematics (c.f. Behr et al., 1992; Hilton et al., 2013; Lamon, 2007), a test followed up with a clinical interview was designed. The clinical interview method is suitable for sufficient analysis of individual results (Piaget, 1997). It serves three aims: "[...] the discovery of cognitive activities (structures, processes, thought patterns, etc.) the identification of cognitive activities, and the evaluation of levels of competence." (Ginsburg, 1981, p. 5)

Data consist of test results from 32 students, with three students' results together with the clinical interview being in focus. Thus, the subject in paper 1 is the multiple cases (de Vaus, 2001) of the three students taking the detection test and their following up clinical interview. In Thomas' (2011) characterization, a subject can further be selected and classified in three ways: a key case, a local knowledge case or an outlier case. The three students in study 1 constitute *key cases*, that is, the group of students are selected because of their particular relevance for the specific phenomenon of identifying students' prior knowledge on proportional reasoning.

The object of the study constitutes the analytical or theoretical frame, under which the research is carried out (Thomas, 2011). The multiple cases with the answers from three students, the test's feasibility to capture students' prior knowledge of proportional reasoning is the object of study. Furthermore, to frame the research approach the typology from George and Bennett (2005) can be used to classify the study as a *theory testing* case study, assessing the validity and scope conditions of using proportional reasoning to analyze students' prior knowledge.

In paper 2, an adult student's motivation to return to studies is investigated in depth. Since we know that adult students' motivation to return to studies vary (c.f. Swain et al., 2005; Strässer & Zevenbergen, 1996), this may impact their faith in their ability to follow through. The possibility to operationalize Mellin-Olsen's (1981, 1987) educational concept of rationales for learning is investigated. Data consist of three semi-structured interviews with one student in the Swedish prison education program. The subject of this *single key case* study (Thomas, 2011) was the student's mathematical experience. The student's mathematical journey is used to try the feasibility of the object; the possibility of using the theoretical concepts S-, and I-rationale to explain the student's motivation, and how it changes. The study also (like in paper 1) represents a *theory testing case study* (George & Bennett, 2005), where the validity and scope conditions of the S-, and I-rationale are tested as a tool for practical inquiry.

Paper 3 represent a *single outlier case*, chosen for its manifestation of particular knowledge of the analytical object of the inquiry (Thomas, 2011). Outlier cases constitute special events and are chosen for their difference to manifest particular knowledge of the analytical object of the inquiry. The study

describes a retrospective reflection of a chain of events with a student, which inspired an analysis of possible causes and effects. The background is that students in the prison education program often have a history of failure as mathematics learners. This may cause difficulties when giving feedback on students' assignments since their negative feelings may put them in a defensive state. The subject of the study was the teacher and student interaction over time, and how it changed as the teaching changed. Data consisted in a retrospective reflection on events. The object of study was how the timing of feedback impacted the interaction. This outlier case can further be classified as a *heuristic case study* (George & Bennett, 2005), wherein new causal paths for how to give feedback to students could be identified. For this type of investigations outlier cases may be especially valuable (George & Bennett, 2005).

In paper 4, the subject was a student's choice of schemes to solve a problem. It was used to explain how the language representation impacts the actions leading to a representative error of a problem, which is the object of the study. Data consisted of a student's written solution to the problem. The student's solution was a *key case* because it represented a common class of error. But it was also a *local knowledge case* where the researcher's familiarity with the case is a prerequisite for the interest to investigate the phenomenon. This is particularly relevant for researchers working from the inside, researching his or her own practice. One of the benefits will be intimate knowledge and ample opportunity for informed, in-depth analysis. Also, the local knowledge of what kind of speed problems the student had earlier shown to be capable of solving was important for the analysis of the student's solution. That knowledge was dependent on my double role of also being a teacher-researcher. The reason that the typical error could be identified was given by the study presented in paper 1. In George and Bennet's (2005) typology the study represents a *disciplined configurative case study*, where the established theory of schemes and representations, from Vergnaud (1998a), were used to explain the case.

Table 1. Case study methodology for the four papers in the thesis.

Paper	Subject	Case	Object	Type of research	Data
1	Three students taking the test and interview	Multiple key cases	The feasibility of the test	Theory testing, assessing the validity of using proportional reasoning to analyse students' prior knowledge	32 test results and three interviews
2	A student's mathematical experience over time	Key	Rationales for learning as a method of analysis	Theory testing	Three Interviews with one student on different timepoints
3	The narrative of a student and a teacher's interaction over time	Outlier	How the timing of feedback impacted the interaction	Heuristic	Retrospective analysis of events in feedback situations
4	A student's choice of schemes when solving a particular problem	Key/local knowledge	The effect of linguistic representation in a problem formulation	Disciplined configurative	A written solution from one student

All of these studies are dependent on the fact that I, as a teacher, have inside information which gives me access to the various cases. This, in turn, entails some ethical considerations that are described in the next section.

## 4.2 Ethical Considerations

The main ethical considerations for the studies comprising this thesis concern my double role of being a teacher and a researcher and how that may affect the research. In this section, I first describe benefits and drawbacks of the double roles and thereafter I account for other ethical considerations.

### 4.2.1 The Double Roles of Being a Teacher-Researcher

The investigations reported in this thesis all rely on me fulfilling the dual roles of being both a teacher and a researcher. As such, I have moved in and out of context, taking different perspectives, depending on if I have conducted teaching or research on my practice. A common approach for insider approaches is that they focus on issues of practice, seeking to probe beneath the surface of the obvious and taken for granted. Such research approaches can be framed in different methodologies, such as action research, teacher narratives, teacher research and, as in my research, case studies (Ball, 2000). The common goal

for these approaches of inquiry is to produce knowledge that can contribute to the improvement of teaching and learning.

Research from inside the practice comes with both benefits and pitfalls (Ball, 2000). The boundaries between teaching and research may sometimes seem blurred. The teacher-researcher approach imply that you have first-person approaches to inquiry (Ball, 2000). However, the role of the teacher as a researcher can be questioned because of the inside perspective, thus being a part of the students' context. This is especially complicated when the classroom-practice is the object of study because the person that will analyze the practice is a part of the context to be analyzed. This may lead to information becoming available that will be hard to account in terms of the analytical methods applied. On the other hand, it can be considered to be a strength to have the first-person's bigger picture if you want to understand practice (Lampert, 2000). In line with Wilson (1995) I believe that the teacher as a researcher, rather than being two separated roles, are two roles forming two sides of the same coin:

When I decide to do research on my teaching, I don't enter the classroom one part teacher, one part researcher. I'm Suzanne, moved at once to help students learn and intensely curious about teaching and learning. In the room and in relation with my students, I am teaching. I am also collecting information (journals, videotapes, interview transcripts, fieldnotes, students' work) that can be used in subsequent analyses. All of this work is driven by the same questions: What might it take to help students learn social studies in meaningful ways? Are my students learning? (Wilson, 1995, p. 20)

In this thesis the first-person's inside role was a prerequisite for two of the studies (paper 2 and 3). The phenomena studied would not have been recognized without it. The teacher role supported the researcher role with key cases and an outlier case to investigate. The key case of a student's rationales for studying mathematics (paper 2) and the outlier case, the retrospective analysis of a student's reactions to immediate feedback (paper 3). For the key case presented in paper 2, my teaching was a part of the context that formed the shifts in the student's rationales. If the object of study had been the shift from I- to S-rationale, trustworthiness would have been hard to claim. But, this shift and what elements of teaching that may have caused it was not the object of study. The object of study was instead the possibility of using the theoretical concepts S-, and I-rationale to account for the student's motivation and how it changed over time.

For the case presented in paper 3, in my teacher role, I lacked tools to analyze the situation and understand what mechanism that come into play in the transfer from regular- to distance education. I only noticed the peculiarity of the situation. However, zooming out from the situation opened opportunities for retrospectively analyzing the interaction between the student and me, which opened opportunities for probing deeper in the phenomena. For this

study, being part of the context was a prerequisite for information about the subject, namely the teacher and student interaction over time and how it changed as the context changed. These cases support that first-person approaches are especially beneficial for getting access to and identifying subjects (Thomas, 2011). The teacher role is a prerequisite for identifying the subjects as an instance of some phenomenon that can be used to identify an object which can be analyzed to explain the phenomenon (Thomas, 2011). Since the subject is selected because it is interesting or telling something from which the object can be identified, an insider position is beneficial for the selection of subjects.

For the study presented in paper 1, it is already known that adult students have a large variety in prior knowledge (Gill & O'Donoghue, 2007) and basic concepts (e.g., Doyle et al., 2016; Díez-Palomar et al., 2006). Factors that influence students' prior knowledge is the amount of years away from mathematics learning and the level of achievement reached in their prior education. No specific inside information was necessary to identify this phenomenon. However, to plan and conduct research on individualized instruction in the prison education context, being a part of the context gives valuable information. (Ball, 2000; Thomas, 2011). For example, the three students chosen as key cases for different student profiles that may emerge from the test are identified from my inside, first-person teacher role (paper 1).

To summarize, since I focus on issues of my special practice of individualized instruction, seeking to probe beneath the surface of the obvious and taken for granted, an insider perspective has been beneficial. However, it is of great importance for the trustworthiness to move in and out of the teacher role and the researcher role with awareness of the different characteristics that come with the different roles.

#### 4.2.2 Data Collection

An important part of research ethics concerns questions about how people that are subjects of research may be treated (Vetenskapsrådet, 2017). Subjects and informants shall in the greatest possible way be protected from injury or violation of integrity in connection with their participation in research. The basic individual protection requirement can be specified in four general requirements for research. These requirements are the *information* requirement, the *consent* requirement, the *confidentiality* requirement and the *use* requirement (Vetenskapsrådet, 2002). The basis for research ethics is thus to protect the integrity of the participants. The Swedish Personal Data Act regulates the protection of personal privacy for participants in research projects (SFS 1998:204). Anonymous data is not covered by the Personal Data Act. Data is considered anonymous if it is not possible to trace the person through any stored documents containing personal data (personal communication, Jonas Åkerman, coordinator of research ethics, 2019-08-23). In order for anonymity

to be maintained, the descriptions of the persons in reports of the research must also be so general that they cannot be identified. Since the obligations for researchers in social science are of critical importance I will account for my ethical considerations study by study.

In the first study: *Designing a research-based detection test for eliciting students' prior understanding on proportional reasoning* (Ahl, 2019) (paper 1), the test design was tried out in two steps. First, I conducted a pilot version of the test consisting of 22 items with 12 voluntarily participating mathematics students. The subjects were invited to participate by teachers working at different prisons in the middle of Sweden. The participants were informed that their test result would be anonymous and that they could interrupt their participation at any point without negative consequences. I personally visited the prisons and distributed the test to the participants, again reminding them of their right to, at any time, end their participation without negative consequences. Furthermore, the students were informed that the aim of their participation was only to try out a test aiming to capture students' prior knowledge on proportional reasoning.

The participants took the test without giving any personal information. This step was performed to get information about the clarity of the item's description and how long the test takes to complete. Oral feedback after the test provided the insights that the test was a bit too long and that some of the items were difficult to interpret. In the second step, a revised version consisting of 16 proportional reasoning items was distributed to colleagues of mine during a teacher gathering in May 2016. My colleagues and I invited our ongoing mathematics students to take the test voluntarily, with informed consent. Informed consent means that the researcher has informed the participants about their role in the project and the conditions that apply to their participation. The participants were informed that participation is voluntary and that they have the right to cancel their participation at any time without reprimands. Also, the information included all the elements of the current investigation that could reasonably affect the subjects' willingness to participate (Vetenskapsrådet, 2002). Informed consent is in general preferably given in a written statement, where the subject acknowledges that they have understood the terms of participation and consent to participation. However, in the prison education program we do not store documents containing personal data and the consent was therefore given verbally. A total of 32 students chose to participate. At this stage of development, the aim was to check that the test discriminated among students (see results section). Anonymous test results from the participants were compiled and presented in tabular form. Anonymization or de-identification requires that the connection between answers to the test and one particular individual has been eliminated so that neither an unauthorized person nor the researcher can restore it (Vetenskapsrådet, 2017). Nobody can thus, for example, combine a certain given answer with the identity of a particular individual. Such a procedure for de-identification was followed.

There is one exception from anonymity. My local students. In my role as a teacher I conduct clinical interviews with all students after they have taken the test. While the course is in progress, I save the results of the tests and notes from the interviews as work material in my teaching practice. For the duration of the course, the information is not research data, just a teacher's work material of documentation. It is only when I request permission to use the information for research that I generate research data. Therefore, the research data is only generated at the time point when the subjects give informed consent for me to use it for research. Three cases of different prior knowledge are described in the article. These subjects are anonymized and presented under pseudonyms. They have all given informed consent for me to use them as anonymous cases in reports of the study. This consent was asked for after finishing the course. The consent was given verbally in the presence of a teacher colleague in the prison education program, as written consent was not possible due to the reasons given above.

Another concern was if the students might have felt pressured to participate in the research, due to the teacher-student relation that I had with the three subjects. The displayed prior knowledge is based on test results and notes from the follow up clinical interview. This is an activity that is incorporated into my regular teaching practice and all students are subject to prior knowledge assessment as well as clinical interviews. Therefore, the students behind the cases described in this thesis have not been asked or pushed to do anything other than ordinary students. However, since permission to use the student results, from the test and the following up interview, in the research was not asked for until after the course grading had been done, there was no reason for the students to believe that them giving consent or not would influence their education or grading.

In paper 2 several ethical considerations about the integrity of the key case Bill is of interest. For the same reasons accounted for above, the informed consent from the student was given verbally, recorded during the first of the three interviews. My research collaborator, Ola Helenius, conducted the first interview. He had no previous connections to the student before the first interview. He gave a clear and concise overall description of the project, including its background and purpose. He described the study, aiming to make it clear to the informant what participation meant for the research and clarified that participation was voluntary and could be terminated at any time, without any specific reasons and without any adverse consequences for the student. Both the second and the third interviews were conducted by me, the student's teacher. That the researcher was also the student's teacher may increase the risk that the student feels pressured to participate, so called information bias (Mercer, 2007). However, the second interview was initiated by the student himself, who wanted to talk to me about new insights in relation to the research question and the ongoing course. It is reasonable to believe that the student initiated this talk with no pressure. The third interview was initiated by me,



several months after the student completed his mathematics course, when the teacher-student relation had ceased to apply. The circumstances surrounding the two last interviews thus indicate that any pressure was either minimal or nonexistent. The interviews contained no sensitive personal data, as defined in 13§ of the Personal Data Act (SFS 1998:204).

The third study is called *Distance mathematics education as a means for tackling impulse control disorder: The case of a young convict*, (Ahl, Sanchez, & Jankvist, 2018). Here, there are some dilemmas that required careful ethical considerations. The student himself informed the teachers that he had an impulse control disorder, which is not unusual in the context in which we work. The student gave informed consent for us to make a retrospective analysis of our interaction during the mathematics course. Again, since a practice in prison education is to avoid keeping personal written records, verbal consent to use the student's solution as well as the narrative describing the educational events was obtained in the presence of a teacher colleague.

In the contextual foreground for the study we have the students impulse control disorder which made the feedback situations difficult in face to face instruction. The context thus contains sensitive personal data as defined by 13§ of The Personal Data Act (SFS 1998:204). The Act (SFS 2003:460) on ethical review of research relating to people applies to research involving treatment of personal data referred to in Article 9 (1) of the EU Data Protection Regulation (sensitive personal data), or personal data on violations of law which include crimes, convictions in criminal cases, criminal procedural remedies or administrative detention. The purpose of the law is to protect the individual and to uphold respect for human dignity in research.

The kind of processing of data that is covered by the law is regulated in the regulation (EU) 2016/679 of the European parliament and of the council of 27 April 2016 on the protection of natural persons with regard to the processing of personal data and on the free movement of such data, and repealing Directive 95/46/EC (General Data Protection Regulation) (Regulation, Protection, 2016), article 2:1; and The Personal Data Act 5 § (SFS 1998:204):

This law applies to such processing of personal data which is fully or partially automated. The law also applies to other processing of personal data, if the data is included in or intended to be included in a structured collection of personal data that is available for searching or compiling according to specific criteria.

Thus, the sensitive data in this study does not apply to the conditions stipulated by law, since no fully or partially automated processing has been done. Neither has any structured collection of personal data been made available for searching or compiling according to the specific criteria. Therefore, there was no legal demand to try the ethics through a regional ethics review board. Still, in line with good research practice (Vetenskapsrådet, 2002) there is a need to consider the ethics in foregrounding the students' impulse control disorder.

A first question is if such research is necessary? The answer to this question is yes, impulse control disorder is a relatively common condition in the prison context and knowledge of how to deal with it is important for improving practice. A second question concerns the student's willingness to take part in the research, including foregrounding the impulse control disorder. The student himself was positive to this, and since he is an adult and thus have the right to make decisions for himself, I feel obliged to respect his view. Finally, regarding anonymity, the student is presented under a pseudonym. The student's anonymity is protected since there are no documents with personal data revealing his identity. The only people that could link the subject with the reported research is me, my teacher colleague and the student himself. This statement may be questioned, since the case described contains a large amount of detail and the impulse control condition described may be rare enough for identification in other school settings. However, this is not the case in the prison education program where being a male in his twenties, with impulse control disorder is a common profile. Therefore, I believe that identification solely by the characteristics revealed in the text is not a possibility.

Finally, I want to make a reflection on my choice of naming the student as a convict. Today, I solely use the term 'student' when I communicate my mathematics practice. At the time I wrote the article discussed here, I had still not entirely thought through how my terminology positions myself in relation to my students. Today I avoid naming my students as convicts.

In the fourth study: *The role of language representation for triggering students' schemes*, (Ahl & Helenius, 2018a), I use a student's solution to illustrate a common error. Consent for using the solution and his verbal explanation for his choice of method under a pseudonym was asked for after the course was finished. The consent was given verbally in the presence of a teacher colleague in the prison education program, due to the reasons argued for above. The fact that the course was finished means that there was no longer pressure to give consent due to power relations related to teaching or grading.

## 4.3 Co-authorship

Three of the four studies are made in collaboration with colleagues. All authors are fully responsible for the content of the papers in this thesis. The collaboration follows The European Code of Conduct for Research Integrity (ALLEA, 2017) for good research practice in collaborative working:

- All partners in research collaborations take responsibility for the integrity of the research.
- All partners in research collaborations agree at the outset on the goals of the research and on the process for communicating their research as transparently and openly as possible.

- All partners formally agree at the start of their collaboration on expectations and standards concerning research integrity, on the laws and regulations that will apply, on protection of the intellectual property of collaborators, and on procedures for handling conflicts and possible cases of misconduct.
- All partners in research collaborations are properly informed and consulted about submissions for publication of the research results. (pp. 6–7)

The study reported in paper 2 was made in collaboration with Ola Helenius at the Swedish National Center for Mathematics Education, University of Gothenburg. The idea for the research was mine. The formulation of questions, collection of data, analysis and writing of the final paper was all in collaboration. The division of work for the collection of data and the analysis was approximately fifty – fifty. For the written presentation of the research, my share of the work was greater through setting up a synopsis and doing the major part of the writing. With the same division of labor, before the final version of the paper, we presented the first part of the study in a conference paper for NORMA (Ahl & Helenius, 2017).

The study reported in paper 3 was done in collaboration with Mario Sanchez Aguilar, Legaria, Intituto Politécnico Nacional, Mexico and Uffe Thomas Jankvist, Danish School of Education, Aarhus University, Denmark. The division of work was that I contributed with the case and the data. Mario provided up to date knowledge of the state of affairs for e-learning. Uffe made the synopsis for the writing and facilitated our work. The analysis of the phenomenon was made by the three of us together. The writing was divided between us approximately equally.

The study reported in paper 4 was also made in collaboration with Ola Helenius at the Swedish National Center for Mathematics Education, University of Gothenburg. The idea for the research and the data collection was mine. On the other parts of the research we collaborated on all parts, from formulating questions, analysis and writing the paper. A great deal of the work was reading up on and discussing the scheme concept. Everything we wrote had been discussed over and over again until we could agree on a shared interpretation. The division of labor for the analysis and the writing was approximately fifty – fifty, although the synopsis of the article was mainly my work.

Carrying out the studies as collaborations was instrumental for the quality of research. As described in section 4.2.1, my double roles as a teacher-researcher raised ethical issues about the suitability of interviewing my own students in a researcher role (paper 2). This was circumvented by bringing in an outsider with no previous connections to the interviewee. For the analysis of events in the changing feedback situation, the outsider's eyes and experiences was necessary to make a retrospective analysis, not only based on my insider perception (paper 3). In fact, I could not have isolated the mechanism of the delay of feedback solely based on my own experience. Paper 3 is a result of

several years of collaborative work on getting familiar with, and operationalizing Vergnaud's theories on investigations of concept formation in different school settings. Therefore, it was natural to collaborate in this study too, on the premise that two minds think better than one.

A concluding remark is that it has been truly encouraging and worthwhile for me to collaborate with and learn from more skilled colleagues.

## 5 Summary of Results from the Papers

In this chapter, the research questions presented in section 1.1 are answered. The included papers are summarized, with a focus on the results. Full versions of the four papers are provided at the end of this thesis.

### Paper 1 – Designing a Research-based Detection Test for Eliciting Students’ Prior Understanding on Proportional Reasoning

In the Swedish Prison Education Program only two out of ten students reach a passing grade in their mathematics courses. One factor that impacts the low number of passing grades is the large variation in prior knowledge which makes it difficult for teachers to engage the students at their level. The study reports on a project aiming to enhance students’ opportunities to access mathematics through individualization that accounts for students’ prior knowledge. Research findings on the development of the pervasive mathematical idea of proportional reasoning are used to construct a two-tier test with 16 items on proportional reasoning. The test was designed to work specifically with students with large variation in prior knowledge. The test is combined with a follow-up clinical interview. Students’ ability to reason proportionally is used to elicit students’ prior knowledge, discriminate among students and provide an access point for designing individualized instruction.

The test was designed to have certain properties. Namely, to give information on students’ prior knowledge both *vertically*, in relation to progress throughout school years, and *horizontally*, covering several topics taught in compulsory school. Also, the test should be able to be completed in a limited time and always giving the student an opportunity to give an answer so as to avoid a negative experience of failure when returning to mathematics as an adult. Furthermore, the test has to give information that discriminates among students. Discovering that all students suffer from the same lack of reasoning skills will not help teachers individualize instruction.

In the analysis of 32 test results, it is shown that the test with 16 items discriminate well among students. The results from the written test were

spread between 0 to 13 correct answers. The 16 items test four different aspects of proportional reasoning on different educational levels. The aspects are:

1. The ability to distinguish additive from multiplicative reasoning and recognize when a multiplicative relation is present;
2. The ability to draw connections to the algebraic rules for fractions when working with part/part ratios, part/whole fractions and proportions,  $a:b = c:d$ ;
3. The ability to recognize and use a range of concrete representations for proportions, e.g., tables, graphs, formulas and drawing pictures;
4. The ability to acknowledge the properties of geometrical objects in two- and three dimensions for calculation of scaling and similarity.

To explain how the test, together with the follow-up clinical interview, provided valuable information on the students' prior knowledge three key cases are described that show that the test elicits different profiles on students' prior knowledge on proportional reasoning. The first key case displayed difficulties with the items relating to properties of geometrical objects in two- and three dimensions and calculation of scaling and similarity. For the second key case, the test results showed prior knowledge comparable to the content taught in the middle grades. In contrast to the first described case, whose answers clustered on one key point, this profile of incorrect answers was spread almost equally among the four aspects. The third key case gave correct answers for 5 items out of 16. The follow up clinical interview confirmed that he had trouble understanding the items, not because of problems with the Swedish language, but because of the mathematical written language and its symbols. He said that he considered himself to be a fairly good reader, but whenever a mathematical word appeared in the text he got confused. These insights could be used as an access point for planning individual instruction with a sequence on geometrical properties and scaling in two- and three dimensions.

How can adult students' prior knowledge be identified, in terms of how developed their proportional reasoning skills are? Results show that the test worked in line with the stipulated design goals; namely that during a limited time collect information on different topics and different levels, while always giving the student an opportunity to give an answer. Furthermore, the test showed to discriminate well among students. Different students can be classified in qualitatively different ways with respect to their mathematical

knowledge. This information can be used to design individual instructional sequences in the adult prison mathematics education.

## Paper 2 – Bill’s Rationales for Learning Mathematics in Prison

Students’ motivation for taking up mathematics in the prison education program is often to get a formal qualification. But, the vast majority has bad experiences from previous schooling and in particular from mathematics, which makes their motivation fragile. This study focuses on the applicability of Melin-Olsen’s theoretical constructs the instrumental I-rationale and the social S-rationale as analytical tools when collecting and characterizing information about students’ rationales for learning. From the premise that teaching shall adapt to students, information on students’ rationales for learning can be used to individualize instruction. While the I-rationale represents the reproduction of ideology, culture and the reproductive forces in society, the social S-rationale indicate social importance. It is the rationale for learning evoked in the student by a synthesis of his self-concept, his cognition of school and schooling, and his concept of what is significant knowledge as developed in his social setting.

A key case of one student’s rationales for learning mathematics is described. Data consists of three semi-structured interviews with a student in the Swedish prison education program. From the data, a storyline is constructed, which forms a basis for a chronological narrative of the student’s mathematical journey. The presented key case displays changes of the student’s rationales for learning mathematics. Three interviews were conducted. One interview was conducted during the first part of his mathematics course; the second closer to the end of the course and the third several months after the course was finished.

How can adult students’ motivation for studying mathematics in relation to their social context and their mathematical learning environment be characterized? The analysis shows that the student’s rationales vary in character over time, as a reaction to his educational contexts. The presented key case had a history of not being able to cope with the subject of mathematics in school. In upper secondary school he realized that mathematics was not for him. He felt that mathematics was like “trying to look left and right at the same time”. His I-rationale broke down due to loss of faith in his own capacity. Fifteen years later, when he was serving a prison sentence, he took the opportunity to study mathematics in the prison education program. His I-rationale was working again. But, it was fragile due to his bad experiences from earlier schooling and his S-rationale was weak. During his studies, the S-rationale strengthened and came to override the I-rationale. The S-rationale was driven by a growing wish

to master mathematics. Even after the course was completed, his way of describing his views confirms a sustained S-rationale for mathematics as a subject.

The findings show that you can characterize students' motivation for studying mathematics by means of Mellin-Olsen's educational concepts S-rationale and I-rationale. By identifying students' different rationales opportunities arise for an individualized instructional design. If a student stops learning because both the I- and the S-rationale have ceased to function, then teaching has to build on a revitalization of the S-rationale. In the key case presented, the S-rationale was catered for by using the student's appreciation for words and his interest for history.

Individualized instruction gives opportunities for framing instruction adapted to the student in contrast to offering ready-made teaching sequences and materials to which the student should adapt. The study presented in paper 2 was made on the basis that information on student's motivation is important for this adaption. Results show that the theoretical construct of instrumental (I) and social (S) rationales for learning is useful for understanding a student's changing motivation in relation to the teaching and to the practice of mathematics the teaching entails. The presented key case is an example that shows that it is possible to evoke an S-rational by addressing mathematics connected to a student's interest when you have the opportunity to individualize instruction.

### Paper 3 – Distance Mathematics Education as a Means for Tackling Impulse Control Disorder: The Case of a Young Convict

The study presents an outlier case of student-teacher interaction. In the case reported in this study, feedback did not work in a face-to-face feedback situation. When the teaching changed from a local setting to distance education, the feedback went from not working at all to working well. While distance education is often considered as a means to provide mathematical education to students in remote locations, this paper reports on a case showing that distance education may also be useful in providing mathematical instruction to individuals who are marginalized or disadvantaged, due to their psychological or social conditions. A narrative of the changes in interaction that occurred when a student was transferred from a local student to a distance student is given.

The student had given information that he had an impulse control disorder, which is rather common among students in the prison education program. This condition impacted the feedback situations. The work was organized so that after every solved assignment, he handed in his solutions for feedback. He was



given immediate (apart from the time it takes to write the response) written informative tutoring, simultaneously supported with oral explanations on the written feedback. The purpose was to clarify, and if necessary, deepen the written feedback to avoid misinterpretation. Thus, the first time the student saw the written feedback was together with the teacher. When the teacher explained the written response in more detail, he immediately went into an affective state, defending and arguing that his answers were in fact correct. This led to a communication breakdown instead of informative tutoring. But, the character of the feedback sessions changed when the student was transferred to another prison and became a distance student. Investigating the mechanism behind this change was the aim of the research.

How can the timing of feedback impact adult students with affective feelings towards mathematics? A retrospective analysis of what had happened when the learning environment of the student changed from local to distance education. The important difference was the timing of feedback. In the distance setting, the written informative tutoring feedback was given in an intranet forum. After a day or some days, the teacher had a dialogue with the student about his work, over telephone, giving the same type of explanations and clarifications that was given directly in the previous feedback setting. For the outlier case presented, the distance setup worked well in contrast to the feedback given in the face-to-face situation where written and oral feedback was given at the same time. In the face-to-face situation the student could not listen to feedback without getting into a state of affect. Interestingly though, this did not occur when the written feedback was given in the intranet forum without face-to-face interaction. The transition to distance education made it possible for the student to follow mathematical instruction adapted to a prison environment, which again helped him to modify his attitude towards the study of mathematics. Furthermore, the distance education setting provided him with an environment in which he could control his impulse control disorder related outbursts, originally triggered by the mathematics lessons and the associated feedback processes. Results indicate that distance education has unforeseen potential in terms of mathematical education for learners who are disadvantaged due to their psychological and social conditions. Additionally, a delay between the written feedback and a complementary oral discussion about the strengths and weaknesses in the student's work functioned as a 'roundabout' for avoiding a situation where the student ended up in affect. A delay between written and oral feedback works as a mechanism that gives the receiver time and space to reflect on the feedback.

## Paper 4 – The Role of Language Representation for Triggering Students’ Schemes

Language representation in tasks is pivotal for how students assimilate situations. The study reports on a key case where the language representation triggered two different schemes. A student’s written solution to an item in the test on prior knowledge of proportional reasoning (paper 1) is analyzed by using scheme theory. Through the work of Vergnaud, schemes were connected to representations and theoretical models from Piaget were connected to principal insights from Vygotsky. This theorization was used to analyze a case of an adult student’s work on the situation to the *steep hill* item, involving average speed.

Steep hill: There is a path up a quite steep hill in Athens. Rickard, who is in good shape, is going up the hill in an average speed of 3 km per hour. He goes down in double speed. What is Richard’s average speed for the whole walk?

The analysis shows that, initially, the student interpreted the situation given as a situation about average speed. It was known from earlier work that the student was well acquainted with, and could handle, standard situations about average speed where two quantities are given and the third is missing. However, the complexity of the steep hill problem is higher than the standard situation, because it involves two different average speeds in one journey.

The student’s solution illustrates several concepts-in-action related to speed. All information needed to deal with the situation is represented in his solution. He just had to finalize his calculation by dividing the sum of the up- and downhill distances with the total time. Interestingly, he abandons his schemes and goes for a scheme related to the arithmetic average instead; using the standard algorithm for the arithmetic average, he arrives at the answer 4.5, obtained by adding 3 and 6 and dividing by 2, that is by averaging the two average speeds. His actions show that he favors the arithmetic average scheme, despite being close to a correct solution building on several elements of a speed scheme.

How can the signifying role of language representations for triggering erroneous schemes in situations involving scientific concepts be theorized? When asked, the student pinpoints the word “average” for his choice of solution. Hence, the conclusion is drawn that the word “average speed” triggered two different solution schemes to the same problem. In this key case, the solution shows that the student’s interpretation of the task invokes a number of operational invariants that efficiently mathematizes the speed situation that the item formulation intends to signify. However, the word “average” also triggers a separate meaning, namely that of the arithmetic average. For the phenomena of arithmetic averages, the student has a very effective scheme, involving a

fast and easy algorithm for computation. In the end, this scheme takes over the situation and decides what answer the student produces.

In Vergnaud's theory, what is signified by a word or some other symbol must be mediated through some operational invariant in an individual's scheme. Vergnaud's theory separates an individual's experience of situations and objects in the world, from the individual's use of natural language or other semiotic systems. In the phenomena we extract from the case of Emilé, the symbolic representation in a situation described in the task backgrounds the individuals' interpretation of the situation as a whole. Instead one particular word becomes foregrounded and acts as an identifier for the situation, thus surpassing all other information given in the situation. By identifying exemplary phenomena in the presented key case, it is shown how previous theory connecting schemes and representations can be extended to allow alternative explanations for a well-known class of students' errors.

Results show that the common error, exemplified by one student's solution, is not just any misunderstanding of how to calculate average speeds. It is specifically induced by a linguistic association. We suggest a theoretical extension of Vergnaud's theory on schemes and their relation to representations by detailing the relationship between schemes and semiotics. Results show that linguistic representations in the problem formulation triggers two separate schemes for the student, one associated to the speed concept and one to the arithmetic average. It is shown that one word in the task can be given agency in the interpretation of the whole situation instead of the situation being interpreted through all of the information given in the task.



## 6 Discussion

This thesis is about individualized instruction, defined here as a method of instruction that, while following a fixed curriculum, tailors content and pace to the abilities and interests of each student. The aim of this study was to gain knowledge of how to organize individual mathematics instruction for adult students without an upper secondary diploma in the prison education program, so that they are given opportunities to succeed with their studies and reach their individual goals.

The choice of studies to deal with this aim was made from a problem driven, pragmatic perspective. The context specific problems were identified by me in my teacher role. To carry out the studies and gain new scientific insights it is my role as a researcher that is required. However, for using these insights to improve instruction, the results need to be adapted to practice. The adaptation process again requires access to the teaching context. Access requires inside information. Therefore, one possible way to carry out this adaptation, which relies on both access to research and access to the teaching context, is to take on the double role of being a teacher-researcher. This has been my approach in this thesis.

### 6.1 Revisiting the Use of Theories

The process of adapting research results to tools for individualized instruction calls for a pragmatic use of theories. I have used theory both to provide explanations of some observed phenomena and to provide predictions of certain phenomena. Most importantly, theory has been put to work to give a structured set of lenses through which my observed phenomena have been approached, observed, studied, analyzed and interpreted. As mentioned above, all the studies in this thesis sprung from experienced problems in teaching practice. In that sense they are all problem driven. However, to approach the problems I have used both theory-driven and method-driven questions, which is why the different studies relate to theories in different ways.

The study presented in paper 1, on students' prior knowledge on proportional reasoning, was initially a problem-driven study sprung from my experience from the teaching practice. The test design was based on a conceptual framework (see chapter 3), using accumulated research findings on a specific

mathematical topic. The main goal was not to explain or predict. Instead I deliberately wanted to use previous research on the teaching and learning of proportional reasoning throughout school years. But, for analyzing students' results, I needed an explanatory lens. Such an explanatory lens was provided by: *The theory of conceptual fields* (Vergnaud, 2009). Thus, both a conceptual framework based on the accumulated findings on proportional reasoning as well as a theoretical explanatory lens was used to approach this problem driven question.

To capture students' rationales for learning, the issue dealt with in paper 2, the theoretical concepts from Mellin-Olsen were used as a structured set of lenses. These concepts belong to a complex and coherent theory of mathematics education, with roots in both psychological and socio-political perspectives. The concept of rationales is just one part of this theory and in the study the full complexity of Mellin-Olsen's theory is not used. Rather, the aim was to operationalize the parts of the theory that concerns rationales (Mellin-Olsen, 1981, 1987). Operationalization is here seen as the process of specifying a mechanism for identifying the value of a variable phenomenon. This conceptual framework was used in a methodological approach. So, while the research on students' motivation was initially problem-driven, the study ended up being a method-driven research, trying out the applicability of a conceptual framework. Still, the main purpose was to solve problems in practice and hence it may still be seen as problem-driven research.

In the study concerning feedback to a student that easily got in an affective state of mind, paper 3, a reflection in retrospect on a chain of events in feedback situations sparked an initial analysis and thereafter further collection of data. The phenomena that solved the feedback related problem was identified and scrutinized. This resulted in a practical framework based on the account of the outlier case consisting of the student-teacher interaction. This framework can also be seen as a guide for 'what works', when dealing with students that easily end up in affect in feedback situations. No a priori theory was used.

In the study of an adult student's conceptualization, dealt with in paper 4, a theoretical framework with explanatory power was used. The use of theory here had the role of making visible something that could not be captured without mediation through the theoretical concepts. The results called for an extension and modification of the theoretical framework in use. So, the initial problem that drove the research was an urge to understand what causes students to act in non-effective ways when solving a problem where they had all the knowledge needed to cope with the situation successfully. A theory on schemes and representations was applied as an explanatory lens (Vergnaud, 1998a). Through formulating the question in terms of scheme theory the initial problem-driven study transformed into a theory-driven study. The results from the study led to theoretical implications for how to extend Vergnaud's *Theory of representations* (1998a) to include the case where one word takes the students mind hostage and thereby hands this single word the agency to represent

the entire situation. Yet, the underlying problem-driven urge to improve practice is still at the core of my own rationales.

## 6.2 Revisiting the Use of Methods

In what sense can the results presented in this thesis be generalizable and how valid are the results given the methods applied in the four case studies that make up this thesis? As presented in chapter 4 and in particular in table 1 the methods applied in the four studies are different and the cases play different roles in the study of the research objects. Obviously, results based on analysis of what happens to a single student or a few students cannot be expected to generalize a bigger population of students but generalizations can be made in other regards. A good case narrative can provide irreducible quality and capture phenomena that cannot be captured by quantitative data (Flyvbjerg, 2006). The characteristics of the cases I presented in this thesis would most likely have faded away in a large quantitative data set. However, to capture specific phenomena case study is a necessary and sufficient method for certain research and it holds up well when compared to other methods (Flyvbjerg, 2006).

Let us first consider the validity of the results from the multiple key cases of the three students taking the detection test and the following up clinical interview in paper 1. Thomas (2011) makes the following reflection on the validity of case studies:

—I have argued that the validity of the case study cannot derive from its representativeness since it can never legitimately be claimed to form a representative sample from a larger set. The essence of selection must rest in the dynamic of the relation between subject and object. It cannot rest in typicality. (p. 514)

The three cases in paper 1 were samples from the test results of 32 students. They were chosen to illustrate that the test discriminates well among students, both in relation to the development of proportional reasoning throughout school years and in relation to the four key points (paper 1) for developing proportional reasoning. The relation between the subject, the three students taking the test, and the object, the feasibility of the test, exemplified the variation in prior knowledge.

In what sense, are the results from paper 2 generalizable? In the study of the key case of one student's mathematical experience over time, the object was to operationalize Mellin-Olsen's rationales for learning as a method for analysis of useful information concerning students' motivation for learning mathematics. In this context useful means that it can help a teacher to individualize the teaching in ways which make it more likely for the student to succeed. The remaining question was if it was possible to extract such information

from students in an interview format. The study presented in paper 2 serves as an existence proof of such a possibility (Schoenfeld, 2007). So, while the study presents a narrative of Bill's mathematical experience through eliciting his changing rationales over time, the main result is that it was possible to operationalize the theory.

The study in paper 3 builds on an outlier case of a student-teacher interaction over time. Of all the four studies presented in the thesis, this one builds the least on theory and has the least well-defined method. In essence, the case was initially chosen because of its peculiarities. It was only after discussion among the authors that the object that I focus on, how the timing of feedback impacted the interaction, was found to be an important aspect. Even though the outlier case may appear to be extreme, in many ways the type of phenomena studied and the personality type described is not uncommon in prison education. However, in relation to cases typically displayed in research on feedback and tutoring, the outlier case comes out as unusual. I drew the conclusion that delayed feedback was advantageous for student Andreas. Given the outlier status of the case, the result is not generalizable to general populations. However, in the research this result is tied together with a phenomenon, namely when the feedback itself caused affect and outbursts so severe that the content of the feedback was not taken into consideration. It was in these circumstances that the delayed feedback initially inadvertently caused by the changes in study conditions proved to be a solution. The result was found in a retrospective analysis of the relation between the subject and the object (Thomas, 2011). The validity of this research is therefore dependent on readers recognizing situations that are similar enough that delayed feedback might prove to be a worthwhile solution.

The study presented in paper 4 is methodologically more complex. Here, the case is one student's choice of schemes when solving a particular problem. The case represents a local knowledge case, since I in my teacher role had extensive knowledge of what kind of schemes the student usually applied for solving rate problems. The type of solution that Emilé produced for the problem at hand had been observed in several other situations too (see Ahl & Helenius, 2018b). Therefore, the case was also an example of a key case and it is reasonable to assume that the phenomenon observed in Emilé's solution is common enough to be considered general. The object was the linguistic representation in the problem formulation.

In the thesis, the solution strategy is also generalized in several other dimensions. The single task analyzed concerns the concept of speed. The results are however assumed to hold for similar situations involving other rates. The argument for such a generalizability builds on that it is the same mathematical and psychological reasoning involved when handling any rate as it is when handling speed. Another area of possible generalization concerns theoretical matters related to Vergnaud's theory of representations (1998a). Here, the case



is used to challenge or confirm a theory or framework (Yin, 2014). In the typology used to frame the cases (section 4.1), this key/local knowledge case is categorized as a disciplined configurative case study (George & Bennett, 2005) where the established theory of Vergnaud (1998a) is used to explain the case. Besides explanations, the use of theory gave an empirical example of a case where the semiotic aspects of a situation influence what schemes are applied, which highlights a missing link in Vergnaud's description. While the results build on Vergnaud's theory, the example of Emilé is also used to extend Vergnaud's theory by adding descriptions of the interplay between situations and semiotics.

The process of adapting research results and theoretical tools from case studies resulted in tools for organizing individualized mathematics instruction in the Swedish prison education program. I label the tools as either being *practical tools* or *thinking tools*. Practical tools are defined as ready-made materials and strategies for instruction. The test and the follow-up clinical interview are practical tools for eliciting students' prior knowledge. Also, the strategy to always give feedback with a delay between the written and the oral feedback to avoid affective situations is a practical tool in the form of a teaching principle. This practical tool for instruction came from the insight that a delay between written and oral feedback works as a mechanism that gives the receiver time and space to reflect.

Thinking tools are defined as a structured set of lenses that heightens a person's abilities to identify, think, talk and interpret phenomena. The educational concepts of social- and instrumental rationales are thinking tools for characterizing students' initial rationales as well as a tool for noticing fluctuation in students' rationales in response to individualized assignments. Also, the knowledge that a single word can be given the power to signify the whole situation and thus trigger erroneous schemes works as a thinking tool for analyzing students' solutions in depth. Here, "In depth" refers to an analysis aiming to both identify errors and, even more importantly, identify the causes of the errors. In the next sections I elaborate further on how the tools can be used for individualizing instruction.

## 6.3 Tools for Planning Individualized Instruction

To give students opportunities to succeed with their studies and reach their individual goals, I argue that both prior knowledge and motivation has to be considered when planning individualized mathematics instruction. Adult students enroll in mathematics courses with various prior knowledge and diversity in mathematical background (e.g., Clarke, Ayres, & Sweller, 2005; Gill & O'Donoghue, 2007; Rittle-Johnson, Star, & Durkin, 2009) as well as different motivation for learning (Swain et al., 2005). Consequently, the planning

for individual instruction should be based on students' variation in both prior knowledge and motivation.

The testing of students' proportional reasoning skills has gained a lot of research interest. Studies on children in primary and secondary school confirm that students tend to overuse additive strategies in the early years (Misailidou & Williams, 2003; Van Dooren, De Bock, Hessels, Janssens, & Verschaffel, 2005). Later on, students tend to overuse multiplicative strategies, such as cross-multiplication, on additive problems. While studies of the school years 4 - 8 are quite common (e.g., Ben-Chaim, Fey, Fitzgerald, Benedetto & Miller, 1998; Fernández, Llinares, Van Dooren, De Bock, Verschaffel, 2012; Hilton et al., 2013; Misailidou & Williams, 2003; Taylor & Jones, 2016), studies on adults' reasoning skills for diagnostic reasons are rarer. An exception is Gläser and Riegler's (2014) test designed to gain information about beginning university students' capacity for proportional reasoning. They found that students in a university bridging course were less capable of proportional reasoning than could be expected, considering their level of prior education. Thus, both children, (Misailidou & Williams, 2003; Van Dooren, et al., 2005) and adults (Gläser & Riegler, 2014) tend to apply additive strategies on proportional problems and multiplicative strategies on additive problems. This error could also be seen in the study of students' prior knowledge on proportional reasoning (paper1) in the prison education program. A difference between these studies and my approach is that I both constructed a test and combined it with clinical interviews, the aim being to gain a better understanding of individuals' prior knowledge, e.g., the identification of cognitive activities, such as structures, processes, thought patterns, and the evaluation of levels of competence (Ginsburg, 1981).

The practical tool for planning individual instruction builds on using the test on proportional reasoning, the clinical interview and an analysis of the results based on the theory of conceptual fields (Vergnaud, 2009). Since the test showed to discriminate well among students, the results of the test together with the clinical interview has the potential to reveal important information concerning lack of prior knowledge, which can hinder the student from coping with the assigned course. In the clinical interview the insights from the study described in paper 4 can be used as a thinking tool to identify what operational invariants which are used in students' solutions and which signifiers that may have triggered erroneous schemes. Depending on a student's errors related to the respective different key points and the different levels of difficulty, an individual first assignment can be designed for each student. The clinical interview should also be used to gather information about students' motivation. To frame students' rationales, it is necessary to not only probe students' solution schemes in the interview but also to ask about the students' experiences from mathematics, throughout school, work and everyday life. From these narratives, the students' S- and I-rationales can be characterized.

As a thinking tool to characterize students' rationales for learning Mellin-Olsen's (1981, 1987) educational concepts, the S- and I-rationale was found to be useful. In the clinical interview, students' rationales for learning may be categorized in terms of strong or weak I- and S-rationale for learning. In the beginning of a course, the I-rationale is always present to some degree, since education is voluntary, but it can vary in strength. A general driving force, both for imprisoned and other adult students, is to qualify for higher education and at the same time prove that they can cope with the subject of mathematics. These driving forces have been reported several times: from Greek prison students (Papaioannou et al., 2018), prison students in Flandern (Halimi et al., 2017), and Norway (Diseth et al., 2008). For non-imprisoned adults the same motivation factors have been reported (c.f. Swain et al., 2005; Strässer & Zevenergen, 1996). The desire for qualification represents the I-rationale where the prison stay can be a motivating factor in and of itself. It has been shown that many imprisoned students take the opportunity to do something meaningful while serving a sentence (Costelloe, 2003; Manger et al., 2010). However, this initial I-rationale is usually only enough to get prisoners to enroll in education. Once enrolled in studies, the student's initial motivation needs to be maintained by the teacher's instruction (Costelloe & Langvik, 2011). The S-rationale is not driven by the imprisonment itself. For the S-rationale, a student's relation to his generalized others (Mead, 1934; 1965) is of interest as well as his own perception of mathematics as a subject, himself as a mathematics learner and possible double-binds (Bateson, 1973). The theoretical construct of rationales for learning can be used as a thinking tool for understanding both students' initial motivation and students' changing motivation over time in relation to the teaching and to the practice of mathematics the teaching entails. Consequently, the thinking tool of students' rationales for learning could always be present during instruction to notice any change. However, the first time the characterization of students' rationales come into play is when the initial assignment is designed. The initial assignment work as an access point for bridging the mathematical content to be taught with students' prior knowledge.

While the level of difficulty and mathematical orientation of the initial assignment is determined by students' prior knowledge, the mathematical level is independent of the students' rationales for learning. However, the planning for students with a working I-rationale and weak S-rationale should consider that they might have a fragile confidence in their capacity. Therefore, instead of getting less challenging assignments, the support and scaffolding from the teacher may be scheduled so that the student is not left with the experience that they are alone with their mathematical challenges. Students with weak S-rationale may need feedback more frequently than students with a strong S-rationale. Moreover, I argue that for students with weak S-rationales the adaption of instruction in accordance to the students I- and S-rationales shall cor-

respond to adapting the challenges given so as to not overpower the perseverance that a student has at the time being. This may be done by adapting the presentation of the mathematics so that the mathematics the student encounters becomes more appealing to that particular student, which then strengthens students' motives for persevere in working with the mathematics, even when the level of challenge gets higher.

## 6.4 Tools for Analyzing and Giving Feedback on Students' Work

For the analysis of students' use of schemes in their individualized assignments, the extension of Vergnaud's theory of representation (1998a), presented in paper 4, can be used as a thinking tool. Students' reasoning and capacity to handle procedures is made visible by analyzing which operational invariants are in play in students' work (Vergnaud, 2009). However, the results presented in paper 4 shows that you cannot assume situations as objectively available to everyone. Instead, what matters is how a person conceives the situation. If we approach the inherent concepts in students' assignments from a cognitive point of view we can think of them as the products of mental operations (Thompson, 1994). The situation a person engages in is the person's construction. It cannot be said to be objectively accessible to the person. To distinguish between different interpretations of situations the notion of actor language and observer language may be used. Speaking in actor language means speaking for yourself, while speaking in observer language means speaking for another or others (MacKay, 1969). A situation to an observer cannot be taken as the situation to an actor (Thompson & Sfard, 1994). Different actors (even the same actor at different moments) can conceive it in different ways depending on how the actor constructs quantities and relationships he sees as inherent in the situation. Despite what the task constructor had in mind when he created the situation, we cannot control what it will become when understood by a student. So, there is an objective intention from the designer, but situations come to life in subjective ways when understood by an individual. This reasoning is coherent with Maturana's dictum, "Everything said is said by an observer" (Maturana, 1988, p. 27). Consequently, the same words and sentences that form a mathematical task can lead students to construct quite different situations and thus also apply different operational invariants to deal with the situation. That is why a careful analysis of students work includes considering the solutions in relation to how the student may have constructed the situation. Teachers need to continually look for indications that students might have constructed unanticipated, alternative meanings than the one intended in the assignment (Cobb, 1988).

When the student's work with the individual assignment has been analyzed, the next step in the instructional sequence is to give response to student. Feedback should be given in formative form; which has the possibility to move the learner forward (e.g., Wiliam, 2007). Formative feedback can be defined as information communicated to the learner that is intended to modify his or her thinking or behavior for the purpose of improving learning (Shute, 2008). While summative feedback gives the learner information on if solutions are correct or not, formative feedback embraces different aspects of feedback which impacts the students' future learning by having effects on performance, thinking/knowledge, learning potential, and affect/motivation (Ginsburg, 2009). Feedback is significantly more effective when it provides the student with details on how to improve the answer rather than just indicating whether the work is correct or not (c.f. Bangert-Drowns, Kulik, Kulik, & Morgan, 1991; Pridemore & Klein, 1995; Wiliam, 2007). However, although there are strong advocates for formative feedback there is less of a consensus on the issue of timing.

Earlier works on the timing of feedback are contradictory. While some argue that immediate feedback hinders errors from being consolidated in students' minds (Phye & Andre, 1989), supporters of delayed feedback rely on the interference-perseveration hypothesis (Kulhavy & Anderson, 1972). In short, the hypothesis says that errors are likely to be forgotten and thus cannot interfere with retention. Other researchers have found that feedback enhancing procedural skills should be provided immediately (c.f. Clariana et al., 2000; Corbett & Andersson, 2001; Schroth, 1992), while feedback for development of students' concept formation is more beneficial when delivered with a delay (Butler & Roediger 2008; Clariana et al., 2000; Mullet, Butler, Verdin, von Borries, & Marsh, 2014). The results presented in paper 3 supports the results for delayed feedback on tasks aiming to develop students' concept formation. Even so, the result on timing of feedback and tutoring reported in this thesis may not be comparable with the above reported studies. While the arguments for the contradictory viewpoints on immediate versus delayed feedback are based on cognition, the positive effects of the delayed feedback in paper 3 is a result of circumventing situations where a student's feelings put him in an affective state. The different intentions should be kept in mind when comparing results on timing.

Since affective feelings is such a salient feature of prison mathematics education, the results from paper 3 leads to a practical tool for how to give feedback: The written formative feedback could be given to the student followed by a delay before the oral feedback in order to circumvent negative affective feedback situations. It has been shown that adult learners returning to formal education struggle, to a greater extent than children and adolescent learners, with negative affective emotions against mathematics as a subject (Klinger, 2011; Ryan, & Fitzmaurice, 2017; Wedege & Evans, 2006). As phrased by Schlöglmann, (2006, p. 15) "Mathematics in particular is often associated with

negative memories, and so people try to avoid using mathematics in their everyday or vocational lives. This leads to a problematic affective situation in adult-educational mathematics courses.” While of course not all adults have a problematic relation to the subject mathematics, a standard procedure where a delay is routinely implemented between written and oral feedback will benefit anyone suffering from negative affective feelings. At the same time, no one is disadvantaged by having time for reflection before the oral discussion about the student's work. This is why the benefits of formative feedback described above, given with a delay between written and oral feedback, aligns well with the intention to mediate the analysis of student' work in detail to individualize instruction.

I have described how theoretical tools from research have been adapted to a special context as well as compared the results with what is already known. The results, the practical- and thinking tools, can be summarized in a model for individualized mathematics instruction in the Swedish prison education context.

## 6.5 A Model for Individualized Mathematics Instruction

The definition of individualized instruction used in this thesis is instruction that tailors content, instructional technology, and pace to the abilities and interests of each student. It is of course impossible to provide a comprehensive method of instruction that accounts for every possible situation. However, it is possible to describe a model of instruction, where the tools from research come into play by means of being adapted into tools for teaching designed to be functional in the special context of individual instruction in prison education. The results of my work can be formulated in a *Model for Individualized Mathematics Instruction of Adults* (MIMIA). In MIMIA the different aspects investigated in the four studies on which this thesis builds are brought together in an approach to understand how different components of individualized instruction interact.

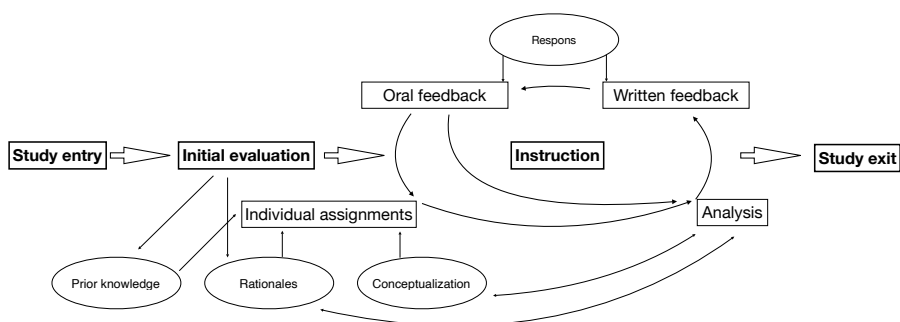


Figure 2. Timeline for working with MIMIA in the prison education program.

In figure 2 the rectangular shapes with boldface represents stages in the course. Oval shapes represent research-based tools that are put to use in the instructional design. In the initial evaluation, the research on prior knowledge and rationales are used as tools to concretely evaluate the student. This evaluation is used as a basis for the instruction phase which is carried out in a cyclic fashion. Specifically, the style and content of an individual assignment is based on the initial evaluation. This individual assignment is the starting point for an instructional cycle. When the student has completed the assignment, it is analyzed using principles of conceptualization research as a tool for thinking. Thereafter, feedback is formulated, which is delivered in written form. As results of the research in this thesis, a delay is then recommended between written feedback and oral discussion with the student. If needed, the student works more on the assignment, and the analysis-feedback stage is repeated. When it is deemed that the student has conceptualized the mathematical content, a new individual assignment is chosen, using the now updated information on the students' knowledge and current rationales. The instructional cycle is repeated until it is considered that the student has conceptualized the full course content sufficiently. The instruction part of the coursework design is therefore cyclic in two dimensions. There is an outer cycle based on different assignments and an inner cycle with analysis, feedback and student work on particular assignments.

MIMIA models a process for individualized mathematics instruction in the Swedish prison education program. The first step, when a new student is enrolled in education, is testing for prior knowledge with the written two-tier test on proportional reasoning. In the follow-up clinical interview, students are given the opportunity to elaborate on their reasoning with some scaffolding from the teacher. In the interview, students' rationales for learning are categorized in terms of strong or weak I- and S-rationales for learning.

Based on the initial information, elicited from the test and the interview, an individualized assignment is designed for each student. Written feedback is delivered, and after a delay followed up by an oral discussion with the student on the strengths and weaknesses in the solutions. The feedback depends on where the student is in the process of solving the problem at hand. The analysis of students' work may eventually cover all relevant aspects that form the basis for grading.

Feedback to students is based on their capacity to communicate their work in the written assignments. But, as the oral feedback is a verbal discussion it also has the role of eliciting further information on what is going on in the students' minds. The information gathered from the analysis of students' work with individual assignments is also used to individualize the next step of instruction.

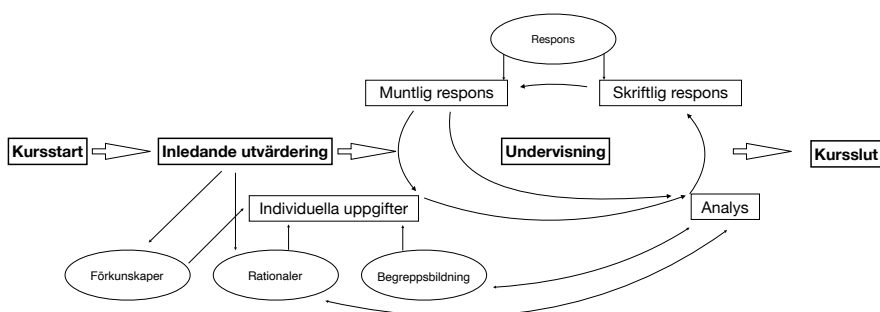
Prison education follows the same regulations as the Swedish municipal adult education (SFS, 2011:1108, 5: 1 §). The Swedish school law (SFS, 2010:800) regulates individualization for municipal adult education in chapter 20: 2§. "Utgångspunkten för utbildningen ska vara den enskildes behov och förutsättningar." [The starting point for the education should be the individual's needs and abilities.]. How this individualization is to be carried out is left for the teachers to figure out. The MIMIA (*Model for Individualized Mathematics Instruction of Adults*) is my contribution to the research field of adults learning mathematics, to be used and adapted to other contexts as one wishes.



# Svensk sammanfattning

## Individualiserad matematikundervisning för vuxna i fängelsekontext

Mina studier av matematikundervisning för vuxna i Kriminalvården har lett fram till en modell för individualisering av undervisning. Modellen synliggör och ger lärare verktyg att hantera viktiga aspekter av individualiseringsprocessen.



MIMIA – En modell för individualiserad matematikundervisning för vuxna i kriminalvårdens vuxenutbildning.

Modellen omfattar två sorters verktyg, praktiska- och tankeverktyg. De praktiska verktygen som ingår i modellen är 1) ett test av förkunskaper och 2) en strategi för respons som gör det möjligt att undvika situationer där den studerande hamnar i negativ affekt vid responstillfället. De tankeverktyg som ingår i modellen handlar om 1) att identifiera drivkrafter för studier och 2) att identifiera hur språkliga representationer i uppgiftsformuleringar påverkar de studerandes begreppsförståelse.

Eftersom undervisning av vuxna innebär speciella utmaningar är det viktigt att studera hur matematikundervisningen kan organiseras så att den ger så bra förutsättningar som möjligt för de studerande att nå sina mål. Vad som särskilt skiljer undervisning av vuxna från undervisning av barn och ungdomar är bland annat en stor variation i förkunskaper och varierande drivkrafter för att återuppta sina studier. Vuxna har också mer negativa känslor för matematik

än barn och ungdomar, eftersom många behöver läsa ämnet just för att de har misslyckats tidigare. Mina undersökningar har gjorts i vuxenutbildningen i Kriminalvården, men MIMIA-modellen kan användas av alla som arbetar med individualiserad matematikutbildning för vuxna.

Med utgångspunkt i att all undervisning av vuxna i Kriminalvården ska vara individualiserad har jag genomfört fyra fallstudier. Studierna handlar om förkunskaper, motivation, respons och begreppsförståelse i den speciella kontexten fängelseundervisning. Tester, intervjuer, interaktion mellan lärare och studerande och skriftliga lösningar från studerande har analyserats i genomförandet av undersökningarna. De fyra studierna har var och en lett fram till ett verktyg i modellen.

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